Exercise 2. We notice that

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle = \langle (A^*A)^{\frac{1}{2}}x, (A^*A)^{\frac{1}{2}}x\rangle = \||A|x\|^2$$

which implies that $\mathrm{Ker}(|A|) = \mathrm{Ker}(A)$ and therefore we may define the linear map $V : \mathrm{Ran}(|A|) \to \mathrm{Ran}(A)$ as

$$V(|A|x) = Ax, \tag{1}$$

for $x \in \mathcal{H}$. Observe that

$$\|V|A|x\| = \|Ax\| = \||A|x\|$$

which means that V is an isometry on $\mathrm{Ran}(|A|)$. We extend V onto $\overline{\mathrm{Ran}(|A|)}$ by continuity and we also set $Vx = 0$ for $x \in (\mathrm{Ran}(|A|))^{\perp}$. Thus, we have that V is an isometry on $(\mathrm{Ker}(V))^{\perp} = (\mathrm{Ker}(A))^{\perp} = (\mathrm{Ker}(|A|))^{\perp} = \overline{\mathrm{Ran}(|A|)}$ and trivially for all $x \in \mathcal{H}$

$$Ax = V|A|x.$$

It is east to check that the adjoint of V is the linear map $V^* : \mathrm{Ran}(A) \to \mathrm{Ran}(|A|)$ given by the expression

$$V^*Ax = |A|x \tag{2}$$

which finishes the proof. □

Exercise 3. We start with the first question.

Let $\{\phi_n\}$ be an ONB of $\mathrm{Ker}(U)^{\perp}$. Since $\mathcal{H} = \mathrm{Ker}(U) \oplus \mathrm{Ker}(U)^{\perp}$ we may extend this ONB to an ONB of the Hilbert space $\mathcal{H}$ by adding vectors $\{\psi_n\} \subset \mathrm{Ker}(U)$. Then by definition we have

$$\mathrm{Tr}(U^*|A|U) = \sum_n \langle \phi_n, U^*|A|U\phi_n\rangle + \sum_n \langle \psi_n, U^*|A|U\psi_n\rangle = \sum_n \langle \phi_n, U^*|A|U\phi_n\rangle \tag{3}$$

since $\{\psi_n\} \subset \mathrm{Ker}(U)$. Now by recalling that U is an isometry on $\mathrm{Ker}(U)^{\perp}$ we may extend the ONB $\{U\phi_n\}$ of $\mathrm{Ker}(U)^{\perp}$ to an ONB of the Hilbert space $\mathcal{H}$ by adding vectors $\{\Phi_n\}$. Hence, by (3) and the fact that $|A|$ is a positive operator we have

$$\mathrm{Tr}(U^*|A|U) = \sum_n \langle U\phi_n, |A|U\phi_n\rangle \leq \sum_n \langle U\phi_n, |A|U\phi_n\rangle + \sum_n \langle \Phi_n, |A|\Phi_n\rangle = \mathrm{Tr}(|A|)$$

which is the desired inequality.

Now we consider the second question.

Using the polar decomposition we may write $A = U|A|$ and $|AB| = V^*AB$ where U and V are partial isometries on $\mathcal{H}$. By the definition we have

$$\|AB\|_{\mathcal{L}^1(\mathcal{H})} = \mathrm{Tr}(|AB|) = \sum_n \langle \phi_n, V^*AB\phi_n\rangle = \sum_n \langle \phi_n, V^*U|A|B\phi_n\rangle = \tag{4}$$

$$\sum_n \langle |A|^{\frac{1}{2}}U^*V\phi_n, |A|^{\frac{1}{2}}B\phi_n\rangle \leq \sum_n \||A|^{\frac{1}{2}}U^*V\phi_n\| \cdot \||A|^{\frac{1}{2}}B\phi_n\| \leq$$

$$\Big(\sum_n \||A|^{\frac{1}{2}}U^*V\phi_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_n \||A|^{\frac{1}{2}}B\phi_n\|^2\Big)^{\frac{1}{2}} = I \cdot II.$$

Term I was estimated above and we have

$$I \leq \operatorname{Tr}(|A|)^{\frac{1}{2}}. \tag{5}$$

To estimate term II first we have to make the following observation: For any bounded linear operator $C : \mathcal{H} \to \mathcal{H}$ and any ONB $\{\phi_n\} \subset \mathcal{H}$ we have the equality

$$\sum_{n=1}^{\infty} \|C\phi_n\|^2 = \sum_{n=1}^{\infty} \|C^*\phi_n\|^2. \tag{6}$$

Indeed, the LHS of (6) equals (we exchange the summation because the summands are positive)

$$\sum_n \sum_m |\langle C\phi_n, \phi_m\rangle|^2 = \sum_m \sum_n |\langle \phi_n, C^*\phi_m\rangle|^2 = \sum_m \|C^*\phi_m\|^2.$$

Hence, with the use of (6) for $C = |A|^{\frac{1}{2}}B$ we estimate

$$II = \Big(\sum_n \|B^*|A|^{\frac{1}{2}}\phi_n\|^2\Big)^{\frac{1}{2}} \leq \|B^*\| \Big(\sum_n \||A|^{\frac{1}{2}}\phi_n\|^2\Big)^{\frac{1}{2}} = \|B\| \cdot \operatorname{Tr}(|A|)^{\frac{1}{2}}. \tag{7}$$

Therefore, (4) with (5) and (7) imply the desired inequality

$$\|AB\|_{\mathcal{L}^1(\mathcal{H})} \leq \|A\|_{\mathcal{L}^1(\mathcal{H})} \|B\|.$$

□

Exercise 4. Consider the polar decomposition of the operator A, i.e. $A = U|A|$ for some partial isometry U on $\mathcal{H}$. As it was done in the proof of Theorem 12.7 we have the following

$$\sum_{n=1}^{\infty} \langle \phi_n, A\phi_n\rangle = \sum_{n=1}^{\infty} \langle |A|^{\frac{1}{2}}U^*\phi_n, |A|^{\frac{1}{2}}\phi_n\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle |A|^{\frac{1}{2}}U^*\phi_n, \psi_m\rangle\langle \psi_m, |A|^{\frac{1}{2}}\phi_n\rangle. \tag{8}$$

Here we have to notice that this double series converges absolutely since by Cauchy-Schwarz inequality we have the upper bound

$$\Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle |A|^{\frac{1}{2}}U^*\phi_n, \psi_m\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle \psi_m, |A|^{\frac{1}{2}}\phi_n\rangle|^2\Big)^{\frac{1}{2}} = I \cdot II.$$

But by applying Parseval's identity and Exercise 3 we have

$$I = \Big(\sum_{n=1}^{\infty} \||A|^{\frac{1}{2}}U^*\phi_n\|^2\Big)^{\frac{1}{2}} \leq \operatorname{Tr}(|A|)^{\frac{1}{2}} < \infty$$

and

$$II \leq \operatorname{Tr}(|A|)^{\frac{1}{2}} < \infty,$$

since $A \in \mathcal{L}^1(\mathcal{H})$. Therefore, we may continue from (8) by exchanging the summation to arrive at

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle |A|^{\frac{1}{2}}\psi_m, \phi_n\rangle\langle \phi_n, U|A|^{\frac{1}{2}}\psi_m\rangle = \sum_{m=1}^{\infty} \langle \psi_m, |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}\psi_m\rangle.$$

In particular (for $\phi_n = \psi_n$) we have shown that

$$\sum_{n=1}^{\infty} \langle \phi_n, A\phi_n \rangle = \sum_{n=1}^{\infty} \langle \phi_n, |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}} \phi_n \rangle. \tag{9}$$

Now starting from the RHS of (9) we obtain (the exchange of summation was justified above)

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle |A|^{\frac{1}{2}} \phi_n, \psi_m \rangle \langle \psi_m, U|A|^{\frac{1}{2}} \phi_n \rangle = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle |A|^{\frac{1}{2}} U^* \psi_m, \phi_n \rangle \langle \phi_n, |A|^{\frac{1}{2}} \psi_m \rangle =$$

$$\sum_{m=1}^{\infty} \langle |A|^{\frac{1}{2}} U^* \psi_m, |A|^{\frac{1}{2}} \psi_m \rangle = \sum_{m=1}^{\infty} \langle \psi_m, U|A| \psi_m \rangle = \sum_{m=1}^{\infty} \langle \psi_m, A\psi_m \rangle$$

and the proof is complete. □

Exercise 5. Let us fix a $g \in \mathcal{L}^1(\mathcal{H})^*$. Observe that for any $\phi \in \mathcal{H}$ the map

$$\psi \mapsto \overline{g(|\phi\rangle\langle\psi|)} \tag{10}$$

is linear in $\psi \in \mathcal{H}$. Thus, by the Riesz representation theorem there exists a unique $u \in \mathcal{H}$ (that depends on ϕ) such that

$$\overline{g(|\phi\rangle\langle\psi|)} = \langle u, \psi \rangle, \text{ for all } \psi \in \mathcal{H}. \tag{11}$$

Equivalently,

$$g(|\phi\rangle\langle\psi|) = \langle \psi, u \rangle, \text{ for all } \psi \in \mathcal{H}. \tag{12}$$

By the uniqueness of u we may define the operator $B : \mathcal{H} \to \mathcal{H}$ as

$$B\phi = u. \tag{13}$$

By the definition of B and (12) we have

$$g(|\phi\rangle\langle\psi|) = \langle \psi, B\phi \rangle, \text{ for all } \psi \in \mathcal{H}. \tag{14}$$

It is straightforward to check that B is a linear operator. For the boundedness of B we have

$$\|B\| = \sup_{\|\phi\|=\|\psi\|=1} |\langle \psi, B\phi \rangle| = \sup_{\|\phi\|=\|\psi\|=1} |g(|\phi\rangle\langle\psi|)| \leq \|g\|_{\mathcal{L}^1(\mathcal{H})^*} < \infty,$$

since $\||\phi\rangle\langle\psi|\|_{\mathcal{L}^1(\mathcal{H})} \leq 1$. To see this, observe that if $A = |\phi\rangle\langle\psi|$ then $A^* = |\psi\rangle\langle\phi|$ and thus, $|A|^2 = A^*A = |\psi\rangle\langle\psi| = |A|$.

It remains to show that the operator $A \mapsto \mathrm{Tr}(BA)$ coincides with g. Wlog by Lemma 12.5 we may assume that A is self-adjoint. Then, by the spectral theorem we may write

$$A = \sum_{n=1}^{\infty} \lambda_n |\psi_n\rangle\langle\psi_n|$$

for some ONB $\{\psi_n\} \subset \mathcal{H}$ and $\{\lambda_n\}$ the corresponding eigenvalues. Since g is continuous we have by (14)

$$g(A) = \sum_{n=1}^{\infty} \lambda_n g(|\psi_n\rangle\langle\psi_n|) = \sum_{n=1}^{\infty} \lambda_n \langle\psi_n, B\psi_n\rangle = \sum_{n=1}^{\infty} \langle\psi_n, BA\psi_n\rangle = \mathrm{Tr}(BA)$$

which finishes the proof.

□