Exercise 6. We start with the first question. Let $\rho \in \mathcal{L}^1(\mathcal{H}^1)$. By Lemma 12.5 wlog we may assume that ρ is self-adjoint. Then by the spectral theorem we may write

$$\rho = \sum_{n=1}^{\infty} \lambda_n |\psi_n\rangle\langle\psi_m|$$

where $\{\lambda_n\}$ are eigenvalues and $\{\psi_n\}$ an ONB of $\mathcal{H}^1$ of eigenvectors. Since $\rho \in \mathcal{L}^1(\mathcal{H}^1)$ we know that

$$\sum_n |\lambda_n| < \infty$$

and so for a given $\epsilon > 0$ we can find $N \in \mathbb{N}$ with the property that

$$\sum_{n=N+1}^{\infty} |\lambda_n| < \frac{\epsilon}{2}. \tag{1}$$

By the definition of P_M we know that for any $\psi \in \mathcal{H}^1$ we have (e.g. Parseval's identity)

$$\lim_{M\to\infty} \|P_M\psi - \psi\| = 0$$

and hence, for all sufficiently large $M \in \mathbb{N}$ we have

$$\|P_M\psi_n - \psi_n\| < \frac{\epsilon}{2N\|\rho\|_{\mathcal{L}^1}}, \text{ for all } n \in \{1, \dots, N\}. \tag{2}$$

Then we obtain

$$\|\rho - P_M\rho\|_{\mathcal{L}^1(\mathcal{H}^1)} = \mathrm{Tr}(|\rho - P_M\rho|) = \mathrm{Tr}\Big(\Big|\sum_n \lambda_n |\psi_n - P_M\psi_n\rangle\langle\psi_n|\Big|\Big) \leq$$

$$\sum_n |\lambda_n| \ \mathrm{Tr}\Big||\psi_n - P_M\psi_n\rangle\langle\psi_n|\Big|$$

but for any $\phi, \psi \in \mathcal{H}^1$ with $\|\psi\| = 1$ we have

$$\mathrm{Tr}||\phi\rangle\langle\psi|| = \mathrm{Tr}\Big[\Big(|\psi\rangle\langle\phi,\phi\rangle\langle\psi|\Big)^{\frac{1}{2}}\Big] = \mathrm{Tr}[(|\psi\rangle\langle\psi|\|\phi\|^2)^{\frac{1}{2}}] = \|\phi\|\mathrm{Tr}[(|\psi\rangle\langle\psi|)^{\frac{1}{2}}] =$$

$$\|\phi\|\mathrm{Tr}(|\psi\rangle\langle\psi|) = \|\phi\|.$$

Therefore, for $\phi = \psi_n - P_M\psi_n$, and by noticing that $\|\psi_n - P_M\psi_n\| \leq 1$ (since $\mathrm{Id} - P_M$ is an orthogonal projection) we can continue our estimate as follows

$$\|\rho - P_M\rho\|_{\mathcal{L}^1(\mathcal{H}^1)} \leq \sum_{n=1}^{N} |\lambda_n| \|\psi_n - P_M\psi_n\| + \sum_{n=N+1}^{\infty} |\lambda_n| < \epsilon$$

for all $M \in \mathbb{N}$ sufficiently large.

For the second question the proof is the same (i.e. wlog we assume that R is self-adjoint and proceed in exactly the same way) but we have to use that $(P_M \otimes \mathrm{Id})\psi \to \psi$ for all $\psi \in \mathcal{H}^1 \otimes \mathcal{H}^2$ as $M \to \infty$. To see this it suffices to consider ψ of the form $\psi_1 \otimes \psi_2$ (by denseness). Then

$$(P_M \otimes \mathrm{Id})\psi - \psi = P_M\psi_1 \otimes \psi_2 - \psi_1 \otimes \psi_2 = (P_M\psi_1 - \psi_1) \otimes \psi_2$$

which implies

$$\|(P_M \otimes \mathrm{Id})\psi - \psi\|_{\mathcal{H}^1 \otimes \mathcal{H}^2} = \|P_M \psi_1 - \psi_1\|_{\mathcal{H}^1} \|\psi_2\|_{\mathcal{H}^2} \to 0$$

as $M \to \infty$ for all $\psi_1 \in \mathcal{H}^1$ by Parseval's identity in the Hilbert space $\mathcal{H}^1$.

□

Exercise 7, Question 1. Let us fix $u, v \in M(R, T)$. We need to show that there are $R, T > 0$ depending only on $\|V\|_\infty$ and $\|\psi_0\|_2$ such that the operator $\mathcal{T}$ satisfies

$$\sup_{-T \le t \le T} \|\mathcal{T}u\|_2 \le R \tag{3}$$

and

$$\sup_{-T \le t \le T} \|\mathcal{T}u - \mathcal{T}v\|_2 \le \frac{1}{2} \sup_{-T \le t \le T} \|u - v\|_2. \tag{4}$$

We start by showing (3). By the definition of $\mathcal{T}$ we have for fixed $t \in [-T, T]$

$$\|\mathcal{T}u\|_2 = \left\| e^{it\Delta}\psi_0(x) - i \int_0^t e^{i(t-\tau)\Delta}[(V * |u|^2)u](\tau, x)\; d\tau \right\|_2 \le$$

$$\|e^{it\Delta}\psi_0\|_2 + \left\| \int_0^t e^{i(t-\tau)\Delta}[(V * |u|^2)u](\tau, x)\; d\tau \right\|_2 =$$

$$\|\psi_0\|_2 + \left\| \int_0^t e^{i(t-\tau)\Delta}[(V * |u|^2)u](\tau, x)\; d\tau \right\|_2 \le$$

$$\|\psi_0\|_2 + \int_0^T \|e^{i(t-\tau)\Delta}[(V * |u|^2)u]\|_2 \; d\tau =$$

$$\|\psi_0\|_2 + \int_0^T \|(V * |u|^2)u\|_2 \; d\tau \le$$

$$\|\psi_0\|_2 + \int_0^T \|V * |u|^2\|_\infty \|u\|_2 \; d\tau \le$$

$$\|\psi_0\|_2 + \|V\|_\infty \int_0^T \||u|^2\|_1 \|u\|_2 \; d\tau = \|\psi_0\|_2 + \|V\|_\infty \int_0^T \|u\|_2^3 \; d\tau,$$

where in the fifth line of estimates we used that $\|FG\|_2 \le \|F\|_\infty \|G\|_2$ (Hölder's inequality) and in the sixth we used that $\|F * G\|_\infty \le \|F\|_\infty \|G\|_1$ (Young's inequality). Let us make the choice

$$R = 2\|\psi_0\|_2. \tag{5}$$

Then in order to prove (3) it suffices to find T such that

$$\|V\|_\infty \int_0^T \|u\|_2^3 \; d\tau \le \frac{R}{2}. \tag{6}$$

But

$$\|V\|_\infty \int_0^T \|u\|_2^3 \; d\tau \le T\|V\|_\infty R^3$$

since $u \in M(R,T)$. Hence, if we have

$$T\|V\|_\infty R^3 \le \frac{R}{2}$$

or equivalently,

$$T \le \frac{1}{2R^2\|V\|_\infty} = \frac{1}{8\|V\|_\infty \|\psi_0\|_2^2} \tag{7}$$

we are done.

Next we show (4). Notice that

$$\mathcal{T}u - \mathcal{T}v = \int_0^t e^{i(t-\tau)\Delta}[(V * |u|^2)u - (V * |v|^2)v]\ d\tau$$

which for fixed $t \in [-T,T]$ implies

$$\|\mathcal{T}u - \mathcal{T}v\|_2 = \Big\| \int_0^t e^{i(t-\tau)\Delta}[(V * |u|^2)u - (V * |v|^2)v]\ d\tau \Big\|_2 \le$$

$$\int_0^T \|e^{i(t-\tau)\Delta}[(V * |u|^2)u - (V * |v|^2)v]\|_2\ d\tau = \int_0^T \|(V * |u|^2)u - (V * |v|^2)v\|_2\ d\tau.$$

Here we have to notice that

$$(V * |u|^2)u - (V * |v|^2)v = (V * |u|^2)(u-v) + (V * (|u|^2 - |v|^2))v = \tag{8}$$

$$(V * |u|^2)(u-v) + [V * (u\overline{(u-v)} + \bar{v}(u-v))]v$$

which leads us to the estimate

$$\|\mathcal{T}u - \mathcal{T}v\|_2 \le \int_0^T \|(V * |u|^2)(u-v)\|_2 + \|[V * (u\overline{(u-v)})]v\|_2 + \|[V * (\bar{v}(u-v))]v\|_2\ d\tau.$$

Again by Hölder and Young inequalities we obtain the upper bound

$$T\|V\|_\infty \sup_{-T\le t\le T} \Big[(\|u\|_2^2 + \|u\|_2\|v\|_2 + \|v\|_2^2)\|u-v\|_2\Big] \le 3T\|V\|_\infty R^2 \sup_{-T\le t\le T} \|u-v\|_2$$

since $u, v \in M(R,T)$. Therefore, if

$$3T\|V\|_\infty R^2 \le \frac{1}{2} \tag{9}$$

the desired estimate (4) is satisfied. But by our choice of T, i.e. by (7) we are done.

Putting (3) and (4) together we have that the operator $\mathcal{T} : M(R,T) \to M(R,T)$ is a contraction and by the Banach contraction theorem we have the existence of a unique $\psi \in M(R,T)$ with the property

$$\mathcal{T}\psi = \psi. \tag{10}$$

Let us also observe that we have the blow up alternative, that is, if T_* is the maximal time of existence then

$$T_* < \infty \ \Rightarrow\ \limsup_{t\to T_*^-} \|\psi(t,\cdot)\|_2 = \infty. \tag{11}$$

This holds because as long as the $L^2(\mathbb{R}^d)$ of the solution ψ stays finite we can repeat the previous argument and extend the solution ψ to a slightly bigger time interval.

For the continuity of the data to solution map we consider $\psi_n, \psi_0 \in L^2(\mathbb{R}^d)$, $n \in \mathbb{N}$, with the property

$$\lim_{n\to\infty} \|\psi_n - \psi_0\|_2 = 0. \tag{12}$$

Let Ψ_n and Ψ be the solutions of the Hartree equation with initial data ψ_n and ψ respectively. Wlog we may assume that for all $n \in \mathbb{N}$ we have $\|\psi_n\|_2 \leq 2\|\psi_0\|_2$. By the previous argument it follows that we may choose a common time of existence T for all the Ψ_n, Ψ and in addition, for all $n \in \mathbb{N}$ we have that Ψ_n, Ψ satisfy

$$\|\Psi_n\|_M,\ \|\Psi\|_M \leq R \tag{13}$$

for some $R > 0$ that depends only on $\|\psi_0\|_2$. Our goal is to show that

$$\lim_{n\to\infty} \|\Psi_n - \Psi\|_M = \lim_{n\to\infty} \sup_{-T\leq t\leq T} \|\Psi_n(t,\cdot) - \Psi(t,\cdot)\|_2 = 0. \tag{14}$$

By the definition of the solutions and with the use of equality (8) (with $u = \Psi_n$ and $v = \Psi$) we may repeat the calculations of the contraction property of the map $\mathcal{T}$ to arrive at

$$\|\Psi_n - \Psi\|_2 = \|\mathcal{T}\Psi_n - \mathcal{T}\Psi\|_2 \leq \|\psi_n - \psi_0\|_2 + \tag{15}$$
$$T\|V\|_\infty(\|\Psi_n\|_M^2 + \|\Psi_n\|_M\|\Psi\|_M + \|\Psi\|_M^2)\|\Psi_n - \Psi\|_M$$

By (13) we can continue as

$$\|\Psi_n - \Psi\|_M \leq \|\psi_n - \psi_0\|_2 + 3T\|V\|_\infty R^2\|\Psi_n - \Psi\|_M. \tag{16}$$

At this point we choose the time T to satisfy

$$3T\|V\|_\infty R^2 < \frac{1}{2}.$$

Hence, from (16) we see the following

$$\|\Psi_n - \Psi\|_M < \|\psi_n - \psi_0\|_2 + \frac{1}{2}\|\Psi_n - \Psi\|_M$$

or equivalently

$$\|\Psi_n - \Psi\|_M < 2\|\psi_n - \psi_0\|_2 \to 0, \text{ as } n \to \infty$$

which completes the proof.

□

Exercise 7, Question 2. First we observe that

$$\|\psi\|_2^2 = \|e^{-it\Delta}\psi\|_2^2 = \left\langle \psi_0 - i\int_0^t e^{-i\tau\Delta}(V*|\psi|^2)\psi\, d\tau, \psi_0 - i\int_0^t e^{-i\tau\Delta}(V*|\psi|^2)\psi\, d\tau \right\rangle \tag{17}$$

from which it follows that the function $t \mapsto \|\psi(t,\cdot)\|_2^2$ is differentiable in time. Therefore, using the Hartree equation and integration by parts we may do the following

$$\frac{d}{dt}\|\psi(t,\cdot)\|_2^2 = 2\text{Re}\langle\psi, \partial_t\psi\rangle = 2\text{Re}\langle\psi, i\Delta\psi - i(V*|\psi|^2)\psi\rangle =$$

$$-2\text{Re } i\langle \nabla\psi, \nabla\psi\rangle - 2\text{Re } i\langle \psi, (V * |\psi|^2)\psi\rangle = 0$$

since both terms are purely imaginary. So the function $\|\psi(t,\cdot)\|_2$ is constant and at 0 is equal to $\|\psi_0\|_2$ which finishes the proof.

The problem with this argument is that we do not know if $\Delta\psi$ makes sense and also we have to justify the integration by parts we have used. Thus, it is better to use (17) and get

$$\frac{d}{dt}\|\psi(t,\cdot)\|_2^2 = \frac{d}{dt}\langle e^{-it\Delta}\psi, e^{-it\Delta}\psi\rangle = 2 \text{ Re}\Big\langle \frac{d}{dt}(e^{-it\Delta}\psi), e^{-it\Delta}\psi \Big\rangle =$$
$$-2 \text{ Re } i\langle e^{-it\Delta}(V * |\psi|^2)\psi, e^{-it\Delta}\psi\rangle = -2 \text{ Re } i\langle (V * |\psi|^2)\psi, \psi\rangle = 0.$$

□

Exercise 7, Question 3. By Question 1 we know that the guaranteed time of existence depends only on the $L^2(\mathbb{R}^d)$ norm of the initial data ψ_0 (and on $\|V\|_\infty$ but V is a fixed function from the very beginning) and by Question 2 we know that the local solution ψ preserves the $L^2(\mathbb{R}^d)$ norm. Thus, from the blow up alternative (11) we must have that the maximal time of existence $T_* = \infty$ which implies that the solution ψ exists globally in time.

□