Exercise 8, Question 1. Let $K_1, K_2, K_3 \in \mathcal{L}^2(\mathcal{H})$, $\lambda \in \mathbb{C}$ and $\{\phi_n\}_{n\in\mathbb{N}}, \{\psi_m\}_{m\in\mathbb{N}}$ be ONB of $\mathcal{H}$.

We start by showing that $\sum_n \langle \phi_n, K_1^* K_1 \phi_n\rangle = \sum_m \langle \psi_m, K_1^* K_1 \psi_m\rangle$. Indeed

$$\sum_{n=1}^{\infty} \langle \phi_n, K_1^* K_1 \phi_n\rangle = \sum_{n=1}^{\infty} \|K_1\phi_n\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle K_1\phi_n, \psi_m\rangle|^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\langle \phi_n, K_1^*\psi_m\rangle|^2 =$$

$$\sum_{m=1}^{\infty} \|K_1^*\psi_m\|^2 = \sum_{m=1}^{\infty} \|K_1\psi_m\|^2 = \sum_{m=1}^{\infty} \langle \psi_m, K_1^* K_1 \psi_m\rangle$$

where at the first equality of the second line we used equation (6) from the solutions of HW sheet 2.

Next, since we want to calculate traces of the form $\mathrm{Tr}(K_1^* K_2)$ for $K_1, K_2 \in \mathcal{L}^2(\mathcal{H})$ we need to know that the quantities

$$\sum_{n=1}^{\infty} \langle \phi_n, K_1^* K_2 \phi_n\rangle$$

are independent of the choice of the ONB $\{\phi_n\}$. By Theorem 12.7 it suffices to show that the operator $K_1^* K_2$ is a trace class operator, i.e. $K_1^* K_2 \in \mathcal{L}^1(\mathcal{H})$. To see this, use the polar decomposition to write $|K_1^* K_2| = V^* K_1^* K_2$ where V is a partial isometry in $\mathcal{H}$. Consider an orthonormal family of vectors $\{u_n\}$ of $(\mathrm{Ker}(V))^{\perp}$ and extend $\{u_n\}, \{Vu_n\}$ to $\{U_n\}, \{V_n\}$, respectively, ONB of $\mathcal{H}$. Then

$$\|K_1^* K_2\|_{\mathcal{L}^1(\mathcal{H})} = \mathrm{Tr}(|K_1^* K_2|) = \sum_{n=1}^{\infty} \langle U_n, V^* K_1^* K_2 U_n\rangle = \sum_{n=1}^{\infty} \langle K_1 V U_n, K_2 U_n\rangle \le$$

$$\Big(\sum_{n=1}^{\infty} \|K_1 V U_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \|K_2 U_n\|^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{\infty} \|K_1 V u_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \|K_2 U_n\|^2\Big)^{\frac{1}{2}} \le$$

$$\Big(\sum_{n=1}^{\infty} \|K_1 V_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \|K_2 U_n\|^2\Big)^{\frac{1}{2}} = (\mathrm{Tr}(K_1^* K_1))^{\frac{1}{2}} (\mathrm{Tr}(K_2^* K_2))^{\frac{1}{2}} < \infty.$$

By definition

$$(1) \qquad |\langle K_1, K_2\rangle_{\mathcal{L}^2(\mathcal{H})}| = |\mathrm{Tr}(K_1^* K_2)| = \Big|\sum_{n=1}^{\infty} \langle \phi_n, K_1^* K_2 \phi_n\rangle\Big| = \Big|\sum_{n=1}^{\infty} \langle K_1\phi_n, K_2\phi_n\rangle\Big| \le$$

$$\sum_{n=1}^{\infty} |\langle K_1\phi_n, K_2\phi_n\rangle| \le \sum_{n=1}^{\infty} \|K_1\phi_n\| \|K_2\phi_n\| \le \Big(\sum_{n=1}^{\infty} \|K_1\phi_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \|K_2\phi_n\|^2\Big)^{\frac{1}{2}} =$$

$$(\mathrm{Tr}(K_1^* K_1))^{\frac{1}{2}} (\mathrm{Tr}(K_2^* K_2))^{\frac{1}{2}} < \infty.$$

Hence, $\langle \cdot, \cdot\rangle_{\mathcal{L}^2(\mathcal{H})}$ maps $\mathcal{L}^2(\mathcal{H}) \times \mathcal{L}^2(\mathcal{H})$ into $\mathbb{C}$.

Now we may check the requirements of the inner product.

$$(2) \qquad \langle K_1, K_2\rangle_{\mathcal{L}^2(\mathcal{H})} = \sum_{n=1}^{\infty} \langle \phi_n, K_1^* K_2 \phi_n\rangle = \sum_{n=1}^{\infty} \langle K_1\phi_n, K_2\phi_n\rangle = \sum_{n=1}^{\infty} \overline{\langle K_2\phi_n, K_1\phi_n\rangle} =$$

$$\overline{\sum_{n=1}^{\infty}\langle\phi_n, K_2^*K_1\phi_n\rangle} = \overline{\mathrm{Tr}(K_2^*K_2)} = \overline{\langle K_2, K_1\rangle_{\mathcal{L}^2(\mathcal{H})}}.$$

(3) $$\langle\lambda K_1 + K_2, K_3\rangle_{\mathcal{L}^2(\mathcal{H})} = \sum_{n=1}^{\infty}\langle\phi_n, (\lambda K_1 + K_2)^* K_3\phi_n\rangle = \sum_{n=1}^{\infty}\langle\phi_n, (\lambda K_1)^* K_3\phi_n\rangle +$$
$$\sum_{n=1}^{\infty}\langle\phi_n, K_2^* K_3\phi_n\rangle = \bar{\lambda}\langle K_1, K_3\rangle_{\mathcal{L}^2(\mathcal{H})} + \langle K_2, K_3\rangle_{\mathcal{L}^2(\mathcal{H})}.$$

(4) $$\langle K_1, K_1\rangle_{\mathcal{L}^2(\mathcal{H})} = \sum_{n=1}^{\infty}\|K_1\phi_n\|^2 \geq 0$$

from which it follows that $\langle K_1, K_1\rangle_{\mathcal{L}^2(\mathcal{H})} = 0$ iff $K_1 = 0$.

Therefore, the quantity $\langle K_1, K_2\rangle_{\mathcal{L}^2(\mathcal{H})}$ is indeed an inner product in $\mathcal{L}^2(\mathcal{H})$ which means that the quantity

(5) $$\|K_1\|_{\mathcal{L}^2(\mathcal{H})} := \sqrt{\langle K_1, K_1\rangle_{\mathcal{L}^2(\mathcal{H})}} = \Big(\sum_{n=1}^{\infty}\|K_1\phi_n\|^2\Big)^{\frac{1}{2}}$$

is a norm in $\mathcal{L}^2(\mathcal{H})$.

□

Exercise 8, Question 2. Assume that $K \in \mathcal{L}^1(\mathcal{H})$. By the definition of the norm in $\mathcal{L}^2(\mathcal{H})$ we have to look at the operator $|K|^2 = K^*K$. Since $|K|$ is self adjoint we may write

$$|K| = \sum_{n=1}^{\infty}\lambda_n\ |\phi_n\rangle\langle\phi_n|$$

where $\lambda_n \geq 0$ for all $n \in \mathbb{N}$ and $\{\phi_n\}$ an ONB in $\mathcal{H}$ of eigenfunctions of $|K|$. Then

$$\|K\|^2_{\mathcal{L}^2(\mathcal{H})} = \mathrm{Tr}(K^*K) = \sum_{n=1}^{\infty}\langle\phi_n, |K|^2\phi_n\rangle = \sum_{n=1}^{\infty}\langle|K|\phi_n, |K|\phi_n\rangle = \sum_{n=1}^{\infty}|\lambda_n|^2 \leq$$
$$\Big(\sum_{n=1}^{\infty}|\lambda_n|\Big)^2 = [\mathrm{Tr}(|K|)]^2 = \|K\|^2_{\mathcal{L}^1(\mathcal{H})}$$

which finishes the proof.

□

Exercise 9. We will use the duality $(\mathcal{L}^1(\mathcal{H}))^* = \mathcal{L}(\mathcal{H})$ which was proved in HW2 Exercise 5. We have

(6) $$\mathrm{Tr}(|\rho|) = \|\rho\|_{\mathcal{L}^1(\mathcal{H}^1)} = \sup_{b\in\mathcal{L}(\mathcal{H}^1), \|b\|=1}|\mathrm{Tr}(b\rho)| = \sup_{b\in\mathcal{L}(\mathcal{H}^1), \|b\|=1}|\mathrm{Tr}((b\otimes \mathrm{Id})R)| \leq$$
$$\sup_{B\in\mathcal{L}(\mathcal{H}^1\otimes\mathcal{H}^2), \|B\|=1}|\mathrm{Tr}(BR)| = \|R\|_{\mathcal{L}^1(\mathcal{H}^1\otimes\mathcal{H}^2)} = \mathrm{Tr}(|R|)$$

and the proof is complete.

□

Exercise 10. Let $k \in \{1,\dots,N\}$. The proof is exactly the same as the one presented in class up to the following observation

$$1 - \mathrm{Tr}\Big(\gamma_N^k |\psi^{\otimes k}\rangle\langle\psi^{\otimes k}|\Big) = 1 - \mathrm{Tr}\Big(|\psi_N\rangle\langle\psi_N|\ p_1^\psi \dots p_k^\psi\Big)$$

since $p_j^\psi = \mathrm{Id}^{\otimes(j-1)} \otimes |\psi\rangle\langle\psi| \otimes \mathrm{Id}^{\otimes(N-j)}$ for all $j \in \{1,\dots,N\}$. Then the RHS equals

$$\langle\psi_N,\psi_N\rangle - \langle\psi_N, p_1^\psi \dots p_k^\psi \psi_N\rangle = \langle\psi_N, (1 - p_1^\psi \dots p_k^\psi)\psi_N\rangle =$$

$$\Big\langle\psi_N, \Big([1-p_1^\psi] + [p_1^\psi - p_1^\psi p_2^\psi] + [p_1^\psi p_2^\psi - p_1^\psi p_2^\psi p_3^\psi] + \dots + [p_1^\psi \dots p_{k-1}^\psi - p_1^\psi \dots p_k^\psi]\Big)\psi_n\Big\rangle =$$

$$\langle\psi_N, q_1^\psi\psi_N\rangle + \langle\psi_N, p_1^\psi q_2^\psi \psi_N\rangle + \langle\psi_n, p_1^\psi p_2^\psi q_3^\psi\psi_N\rangle + \dots + \langle\psi_N, p_1^\psi \dots p_{k-1}^\psi q_k^\psi \psi_N\rangle \le$$

$$\sum_{j=1}^k \langle\psi_N, q_j^\psi\psi_N\rangle = k\ c_N(\psi_N,\psi)$$

which is exactly what we need. For the inequality we used that the p_j^ψ are all orthogonal projections and for the last equality we used that $\psi_N \in L_s^2(\mathbb{R}^{3N})$.

□

Exercise 11, Question 1. To see that the expression

$$\Sigma := \sum_{j=1}^N \psi_0^{\otimes(j-1)} \otimes \dot\psi \otimes \psi_0^{(N-j)}$$

is a tangent vector it suffices to observe that

$$(\psi_0 + t\dot\psi)^{\otimes N} = \psi_0^{\otimes N} + t\ \Sigma + O(t^2)$$

which implies

$$\frac{d}{dt}(\psi_0 + t\dot\psi)^{\otimes N}\Big|_{t=0} = \Sigma$$

and therefore, since $\dot\psi \perp \psi_0$ we obtain ($\|\psi_0 + t\dot\psi\|_2 = \sqrt{1+t^2}$)

$$\frac{d}{dt}\Big(\frac{\psi_0 + t\dot\psi}{\sqrt{1+t^2}}\Big)^{\otimes N}\Big|_{t=0} = \Sigma.$$

For the converse let γ be a curve as in the definition of the tangent space $T_{\psi_0^{\otimes N}}$. Using that $\gamma(0) = \psi_0$ and the product rule for the derivative we see that

$$\frac{d}{dt}(\gamma(t)^{\otimes N})\Big|_{t=0} = \sum_{j=1}^N \gamma(0)^{\otimes(j-1)} \otimes \frac{d}{dt}(\gamma(t))\Big|_{t=0} \otimes \gamma(0)^{\otimes(N-j)}.$$

Finally, since $\|\gamma(t)\|_2 = 1$ for all t we obtain (by differentiating in t) that $\dot\gamma(0) \perp \gamma(0) = \psi_0$.

□

Exercise 11, Question 2. By the definition we have

$$\Big\langle V_{12}\psi_0^{\otimes 2}, \phi\otimes\psi_0\Big\rangle_{L^2(\mathbb{R}^6)} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \overline{V_{12}(x_1,x_2)\psi_0(x_1)\psi_0(x_2)}\ \phi(x_1)\psi_0(x_2)\ dx_2dx_1 = \tag{7}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \overline{V(x_1-x_2)}\bar{\psi_0}(x_2)\psi_0(x_2)\bar{\psi_0}(x_1)\ \phi(x_1)\ dx_2dx_1 =$$

$$\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} \overline{V(x_1-x_2)\psi_0(x_2)}\psi_0(x_2)\ dx_2\Big)\bar{\psi_0}(x_1)\phi(x_1)\ dx_1 =$$

$$\int_{\mathbb{R}^3} \bar{V} * |\psi_0|^2(x_1)\bar{\psi_0}(x_1)\phi(x_1)\ dx_1 = \int_{\mathbb{R}^3} [\overline{(V*|\psi_0|^2)\psi_0}](x_1)\phi(x_1)\ dx_1 =$$

$$\Big\langle (V*|\psi_0|^2)\psi_0, \phi\Big\rangle_{L^2(\mathbb{R}^3)}$$

which is the desired equality.

□

Exercise 11, Question 3. For the projection of $-iH_N\psi_0^{\otimes N}$ onto $T_{\psi_0^{\otimes N}}$ we have to calculate

$$\Big\langle -iH_N\psi_0^{\otimes N}, T_{\psi_0^{\otimes N}}\phi\Big\rangle \tag{8}$$

for $\phi\perp\psi_0$. From the definition of the Hamiltonian H_N and symmetry considerations we may rewrite (8) as

$$\Big\langle i\sum_{j=1}^{N}\Delta_{x_j}\psi_0^{\otimes N} - i\alpha\sum_{1\le i<j\le N} V(x_i-x_j)\psi_0^{\otimes N}, T_{\psi_0^{\otimes N}}\phi\Big\rangle = \tag{9}$$

$$iN\Big\langle(\Delta_{x_1}\psi_0)\otimes\psi_0^{\otimes(N-1)}, T_{\psi_0^{\otimes N}}\phi\Big\rangle - i\alpha\ \frac{N(N-1)}{2}\Big\langle V(x_1-x_2)\psi_0^{\otimes(N-1)}, T_{\psi_0^{\otimes N}}\phi\Big\rangle.$$

Since $\phi\perp\psi_0$ we know from Question 1 that the vector $T_{\psi_0^{\otimes N}}\phi$ is given by

$$\sum_{j=1}^{N}\psi_0^{\otimes(j-1)}\otimes\phi\otimes\psi_0^{(N-j)}$$

which allows us to continue the calculation even further and arrive at ($\|\psi_0\|_2 = 1$)

$$iN\Big\langle\Delta_{x_1}\psi_0\otimes\psi_0^{\otimes(N-1)}, \phi\otimes\psi_0^{\otimes(N-1)}\Big\rangle - i\alpha N(N-1)\Big\langle V(x_1-x_2)\psi_0^{\otimes 2}, \phi\otimes\psi_0\Big\rangle =$$

$$N\Big(\Big\langle i\Delta_{x_1}\psi_0, \phi\Big\rangle - \alpha(N-1)\Big\langle i(V*|\psi_0|^2)\psi_0, \phi\Big\rangle\Big).$$

We used the assumption that V is even (that is why the 2 has dissappeared from the denominator of the fraction $\frac{N(N-1)}{2}$) and the equality we showed in Question 2. Hence, we have proved

$$\Big\langle -iH_N\psi_0^{\otimes N}, T_{\psi_0^{\otimes N}}\phi\Big\rangle = N\Big\langle i\Delta_{x_1}\psi_0 - i\alpha(N-1)(V*|\psi_0|^2)\psi_0, \phi\Big\rangle \tag{10}$$

for all $\phi\perp\psi_0$.

Notice that for $\alpha = \frac{1}{N-1}$ the projection of $-\frac{i}{N}H_N\psi_0^{\otimes N}$ onto $T_{\psi_0^{\otimes N}}$ is independent of N.

□

Exercise 11, Question 4. In order to derive the Hartree dynamics we need to identify the $\dot\psi \in L^2(\mathbb{R}^3)$ that are $\dot\psi \perp \psi_0$ with the property that (this is the condition that the difference of a vector and its projection is orthogonal to every other tangent vector)

$$\Big\langle -iH_N\psi_0^{\otimes N} - T_{\psi_0^{\otimes N}}\dot\psi, T_{\psi_0^{\otimes N}}\phi\Big\rangle = 0 \tag{11}$$

for all $\phi \in L^2(\mathbb{R}^3)$ with $\phi \perp \psi_0$. Since $\dot\psi \perp \psi_0$ and $\phi \perp \psi_0$ we obtain

$$\Big\langle T_{\psi_0^{\otimes N}}\dot\psi, T_{\psi_0^{\otimes N}}\phi\Big\rangle = N\Big\langle \dot\psi, \phi\Big\rangle$$

because $\dot\psi$ and ϕ must be in the same component N-times. Thus, (11) becomes

$$N\Big\langle i\Delta_{x_1}\psi_0 - i\alpha(N-1)(V * |\psi_0|^2)\psi_0 - \dot\psi, \phi\Big\rangle = 0 \tag{12}$$

for all $\phi \perp \psi_0$ or equivalently,

$$\Big(i\Delta_{x_1}\psi_0 - i\alpha(N-1)(V * |\psi_0|^2)\psi_0 - \dot\psi\Big) \;//\; \psi_0. \tag{13}$$

Therefore, for $\alpha = \frac{1}{N-1}$ there exists $\lambda = \lambda(\psi_0) \in \mathbb{R}$ such that

$$\dot\psi = i\Delta_{x_1}\psi_0 - i(V * |\psi_0|^2)\psi_0 - i\lambda\psi_0. \tag{14}$$

We can forget the last term of (14) since this is just a change in phase which is only time dependent and hence, since $\dot\psi = \partial_t\psi_t\Big|_{t=0}$ we derive the Hartree equation

$$i\partial_t\psi_t = -\Delta\psi_t + (V * |\psi_t|^2)\psi_t.$$

□