Exercise 12. Let $\epsilon > 0$ be given. From the definition of $\|K\|$ we can find a vector $v \in \mathcal{H}$ such that

$$\|K\|^2 < \epsilon + \|Kv\|^2$$

and $\|v\| = 1$.

Now we extend the set $\{v\}$ to an ONB $\{v_n\}_n$ of $\mathcal{H}$ with $v_1 = v$. Then

$$\|K\|^2 < \epsilon + \|Kv\|^2 \leq \epsilon + \|Kv_1\|^2 + \sum_{n=2}^{\infty} \|Kv_n\|^2 = \epsilon + \sum_{n=1}^{\infty} \|Kv_n\|^2 = \epsilon + \|K\|^2_{L^2(\mathcal{H})}.$$ 

Since this is true for all $\epsilon > 0$ the proof is complete.

Exercise 13. Let $f \in L^2(\mathbb{R}^{3N})$. By definition

$$\|Tf\|^2_{L^2(\mathbb{R}^{3N})} = \int_{\mathbb{R}^{3N}} |Tf(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} dx_j =$$

$$\int_{\mathbb{R}^{3N}} |V(x_1 - x_2)|^2 |f(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} dx_j \leq \|V\|^2_\infty \|f\|^2_{L^2(\mathbb{R}^{3N})}$$

from which we get $\|T\|_{L^2 \to L^2} \leq \|V\|_\infty$.

Similarly, for the operator $S$ the proof is the same if we observe that

$$\left\| V \ast |\psi|^2 \right\|_\infty = \left\| \int_{\mathbb{R}^3} V(x-y) |\psi(y)|^2 \, dy \right\|_\infty \leq \|V\|_\infty \|\psi\|_{L^1} = \|V\|_\infty \|\psi\|^2_{L^2} = \|V\|_\infty.$$ 

Exercise 14, Question 1. Let $\{e_n\}_n$ be an ONB of $L^2(\mathbb{R}^d)$. If we set

$$e_{mn}(x,y) = e_m(x) e_n(y)$$

we obtain the ONB $\{e_{mn}\}_{m,n}$ of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. For fixed $m, n \in \mathbb{N}$ the function $k \cdot e_{mn} \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ since by Hölder’s inequality

$$\|k \cdot e_{mn}\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = \int_{\mathbb{R}^d \times \mathbb{R}^d} |k(x,y) e_m(x) e_n(y)| \, dxdy \leq$$

$$\|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \|e_m\|_{L^2(\mathbb{R}^d)} \|e_n\|_{L^2(\mathbb{R}^d)} = \|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} < \infty.$$ 

Hence, Fubini’s theorem allows us to do the following

$$\langle k, e_{mn} \rangle_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} = \int_{\mathbb{R}^d \times \mathbb{R}^d} k(x,y) e_m(x) e_n(y) \, dxdy =$$

$$\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k(x,y) e_n(y) \, dy \right) e_m(x) \, dx = \left( \int_{\mathbb{R}^d} k(x,y) e_n(y) \, dy \right) e_m(x).$$ 

Consequently, with the use of Parseval’s identity

$$\|L_k\|^2_{L^2(\mathbb{R}^d)} = \sum_{n=1}^{\infty} \|L_k e_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle L_k e_n, e_m \rangle|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle k, e_{mn} \rangle|^2 = \|k\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}.$$ 

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Thus, the operator $L_k$ is a Hilbert-Schmidt operator with norm $\|L_k\|_{L^2(\mathbb{R}^d)} = \|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$. Notice that this calculation shows that the map $k \mapsto L_k$ is an isometry of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ into $L^2(L^2(\mathbb{R}^d))$.

\[\]

Exercise 14, Question 2. Suppose that $T$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$ and let \{\(e_n\)\} be an ONB of $L^2(\mathbb{R}^d)$. We also define \{\(e_{mn}\)\} as in (1).

Since $T$ is Hilbert-Schmidt we have that the sequence \{\(|e_m, Te_n|\)\} is square summable and since \{\(e_{mn}\)\} is an ONB of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ we have that the series

\[
k := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} e_{mn}
\]

converges unconditionally in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, i.e. it converges regardless of the ordering we impose on the index set $\mathbb{N} \times \mathbb{N}$. Let $L_k$ be the integral operator whose kernel is $k$. From Question 1 we know that $L_k$ is a Hilbert-Schmidt operator and $\|L_k\|_{L^2(L^2(\mathbb{R}^d))} = \|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$.

We will prove that $T = L_k$.

From Question 1 we know that the linear map $S : L^2(\mathbb{R}^d \times \mathbb{R}^d) \to L^2(L^2(\mathbb{R}^d))$ given by $S(k) := L_k$ is an isometry. Since unconditional convergence is preserved by continuous linear maps we have

\[
L_k = S(k) = S\left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} e_{mn} \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} L e_{mn}
\]

where the last series in (3) converges unconditionally in $L^2(L^2(\mathbb{R}^d))$. From Exercise 12 we know that the Hilbert-Schmidt dominates the operator norm and therefore, the last series in (3) converges unconditionally with respect to the operator norm. Hence, for each function $f \in L^2(\mathbb{R}^d)$ we have

\[
L_k f = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} L e_{mn} f
\]

where this series converges unconditionally in the $L^2(\mathbb{R}^d)$ norm. By observing that $L e_{mn}$ is the rank 1 operator $L e_{mn} f = \langle e_n, f \rangle e_m$ and using the unconditional convergence to reorder the summations we obtain

\[
L_k f = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} \langle e_{mn}, f \rangle e_m = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle \langle e_n, f \rangle e_m = \sum_{n=1}^{\infty} \langle e_n, f \rangle \left( \sum_{m=1}^{\infty} \langle e_m, Te_n \rangle e_m \right) = \sum_{n=1}^{\infty} \langle e_n, f \rangle T e_n = T \left( \sum_{n=1}^{\infty} \langle e_n, f \rangle e_n \right) = T f
\]

where at the second to last equality we used that the operator $T$ is continuous.

Thus, $L_k f = T f$ for every $f \in L^2(\mathbb{R}^d)$.\]
Exercise 14, Question 3. Question 1 shows that the operator $S : L^2(\mathbb{R}^d \times \mathbb{R}^d) \to L^2(L^2(\mathbb{R}^d))$ given by $S(k) := L_k$ is an isometry and Question 2 shows that $S$ is surjective. Therefore, the proof is complete.

\[\square\]

Exercise 14, Question 4. We present two approaches which are non-rigorous proofs of the desired equality.

**Approach 1:** By definition
\[
\text{Tr}(T) = \sum_{n=1}^{\infty} \langle e_n, Te_n \rangle_{L^2(\mathbb{R}^d)} = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} e_n(x) Te_n(x) \, dx = \\
\sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e_n(x) k(x,y) e_n(y) \, dy \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x,y) \sum_{n=1}^{\infty} e_n(x)e_n(y) \, dy \, dx = \\
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x,y) \delta(x-y) \, dy \, dx.
\]

Here we used that the kernel of the Identity operator $\text{Id} = \sum_n |e_n\rangle\langle e_n|$ is the distribution $k(x,y) = \delta(x-y)$.

**Approach 2:** From expression (2) we have the following
\[
\int_{\mathbb{R}^d} k(x,x) \, dx = \int_{\mathbb{R}^d} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} e_{mn}(x,x) \, dx = \\
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} e_m(x)e_n(x) \, dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle e_m, Te_n \rangle_{L^2(\mathbb{R}^d)} \langle e_n, e_m \rangle_{L^2(\mathbb{R}^d)} = \\
\sum_{n=1}^{\infty} \langle e_n, Te_n \rangle_{L^2(\mathbb{R}^d)} = \text{Tr}(T).
\]

The assumption that $k$ is continuous is to avoid the difficulty of defining the $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ function $k$ on the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ which is of measure 0.

\[\square\]

Exercise 14, Question 5. Since $\rho \in L^1(L^2(\mathbb{R}^d))$ we have that $\rho$ is a Hilbert-Schmidt operator in $L^2(\mathbb{R}^d)$ and from Question 2 we know that it has a kernel function which we denote by $\rho(x,v), x,v \in \mathbb{R}^d$ (i.e. $\rho = L_\rho$ with a small abuse of notation). From the definition of the partial trace we know that $\rho$ is the unique operator in $L^1(L^2(\mathbb{R}^d))$ that satisfies

\[\text{Tr}(k\rho) = \text{Tr}((k \otimes \text{Id})R)\]

for all $k \in \text{Com}(L^2(\mathbb{R}^d))$. Since finite rank operators are dense in $(\text{Com}(L^2(\mathbb{R}^d)), \|\cdot\|_{\text{op}})$ and since finite rank operators are Hilbert-Schmidt operators (every Hilbert-Schmidt operator is also a compact operator) we obtain that the set $L^2(L^2(\mathbb{R}^d))$ is dense in the Banach space $(\text{Com}(L^2(\mathbb{R}^d)), \|\cdot\|_{\text{op}})$. Hence, in (5) it suffices to consider $k \in L^2(L^2(\mathbb{R}^d))$. But then with
the use of Question 2 and Question 4 we may rewrite (5) as (again with a slight abuse of notation we will use $k(x, z)$ to denote the kernel function of the operator $k$)

\begin{equation}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x, z) \rho(z, x) \, dx \, dz = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x, z) \left[ \int_{\mathbb{R}^m} R((z, y), (x, y)) \, dy \right] \, dx \, dz
\end{equation}

or equivalently,

\begin{equation}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x, z) \left[ \rho(z, x) - \int_{\mathbb{R}^m} R((z, y), (x, y)) \, dy \right] \, dx \, dz = 0
\end{equation}

for all $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Therefore, we must have

\[
\rho(z, x) = \int_{\mathbb{R}^m} R((z, y), (x, y)) \, dy
\]

which is the desired equality. \hfill \square