Exercise 12. Let $\epsilon > 0$ be given. From the definition of $\|K\|$ we can find a vector $v \in \mathcal{H}$ such that

$$\|K\|^2 < \epsilon + \|Kv\|^2 \text{ and } \|v\| = 1.$$

Now we extend the set $\{v\}$ to an ONB $\{v_n\}_n$ of $\mathcal{H}$ with $v_1 = v$. Then

$$\|K\|^2 < \epsilon + \|Kv\|^2 \leq \epsilon + \|Kv_1\|^2 + \sum_{n=2}^{\infty} \|Kv_n\|^2 = \epsilon + \sum_{n=1}^{\infty} \|Kv_n\|^2 = \epsilon + \|K\|^2_{\mathcal{L}^2(\mathcal{H})}.$$

Since this is true for all $\epsilon > 0$ the proof is complete.

□

Exercise 13. Let $f \in L^2(\mathbb{R}^{3N})$. By definition

$$\|Tf\|^2_{L^2(\mathbb{R}^{3N})} = \int_{\mathbb{R}^{3N}} |Tf(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} dx_j =$$

$$\int_{\mathbb{R}^{3N}} |V(x_1 - x_2)|^2 |f(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} dx_j \leq \|V\|^2_{\infty} \, \|f\|^2_{L^2(\mathbb{R}^{3N})}$$

from which we get $\|T\|_{L^2 \to L^2} \leq \|V\|_{\infty}$.

Similarly, for the operator S the proof is the same if we observe that

$$\left\| V * |\psi|^2 \right\|_{\infty} = \left\| \int_{\mathbb{R}^3} V(x-y) |\psi(y)|^2 \, dy \right\|_{\infty} \leq \|V\|_{\infty} \||\psi|^2\|_{L^1} = \|V\|_{\infty} \|\psi\|^2_{L^2} = \|V\|_{\infty}.$$

□

Exercise 14, Question 1. Let $\{e_n\}_n$ be an ONB of $L^2(\mathbb{R}^d)$. If we set

$$e_{mn}(x, y) = e_m(x) \overline{e_n(y)} \tag{1}$$

we obtain the ONB $\{e_{mn}\}_{m,n}$ of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. For fixed $m, n \in \mathbb{N}$ the function $k \cdot e_{mn} \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ since by Hölder's inequality

$$\|k \cdot e_{mn}\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = \int_{\mathbb{R}^d \times \mathbb{R}^d} |k(x, y) e_m(x) \overline{e_n(y)}| \, dxdy \leq$$

$$\|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \|e_m\|_{L^2(\mathbb{R}^d)} \|e_n\|_{L^2(\mathbb{R}^d)} = \|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} < \infty.$$

Hence, Fubini's theorem allows us to do the following

$$\left\langle k, e_{mn} \right\rangle_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \overline{k(x, y)} \, e_m(x) \overline{e_n(y)} \, dxdy =$$

$$\int_{\mathbb{R}^d} \overline{\left(\int_{\mathbb{R}^d} k(x, y) e_n(y) \, dy \right)} e_m(x) \, dx = \left\langle L_k e_n, e_m \right\rangle_{L^2(\mathbb{R}^d)}.$$

Consequently, with the use of Parseval's identity

$$\|L_k\|^2_{\mathcal{L}^2(\mathbb{R}^d)} = \sum_{n=1}^{\infty} \|L_k e_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle L_k e_n, e_m \rangle|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle k, e_{mn} \rangle|^2 = \|k\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}.$$

Thus, the operator L_k is a Hilbert-Schmidt operator with norm $\|L_k\|_{\mathcal{L}^2(\mathbb{R}^d)} = \|k\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$. Notice that this calculation shows that the map $k \mapsto L_k$ is an isometry of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ into $\mathcal{L}^2(L^2(\mathbb{R}^d))$.

□

Exercise 14, Question 2. Suppose that T is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$ and let $\{e_n\}_n$ be an ONB of $L^2(\mathbb{R}^d)$. We also define $\{e_{mn}\}_{m,n}$ as in (1).

Since T is Hilbert-Schmidt we have that the sequence $\{\langle e_m, Te_n\rangle\}_{m,n}$ is square summable and since $\{e_{mn}\}_{m,n}$ is an ONB of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ we have that the series

$$k := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Big\langle e_m, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} e_{mn} \tag{2}$$

converges unconditionally in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, i.e. it converges regardless of the ordering we impose on the index set $\mathbb{N} \times \mathbb{N}$. Let L_k be the integral operator whose kernel is k. From Question 1 we know that L_k is a Hilbert-Schmidt operator and $\|L_k\|_{\mathcal{L}^2(L^2(\mathbb{R}^d))} = \|k\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$.

We will prove that $T = L_k$.

From Question 1 we know that the linear map $S : L^2(\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{L}^2(L^2(\mathbb{R}^d))$ given by $S(k) := L_k$ is an isometry. Since unconditional convergence is preserved by continuous linear maps we have

$$L_k = S(k) = S\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Big\langle e_m, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} e_{mn}\Big) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Big\langle e_m, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} L_{e_{mn}} \tag{3}$$

where the last series in (3) converges unconditionally in $\mathcal{L}^2(L^2(\mathbb{R}^d))$. From Exercise 12 we know that the Hilbert-Schmidt dominates the operator norm and therefore, the last series in (3) converges unconditionally with respect to the operator norm. Hence, for each function $f \in L^2(\mathbb{R}^d)$ we have

$$L_k f = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle e_m, Te_n\rangle L_{e_{mn}} f \tag{4}$$

where this series converges unconditionally in the $L^2(\mathbb{R}^d)$ norm. By observing that $L_{e_{mn}}$ is the rank 1 operator $L_{e_{mn}} f = \langle e_n, f\rangle e_m$ and using the unconditional convergence to reorder the summations we obtain

$$L_k f = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle e_m, Te_n\rangle L_{e_{mn}} f = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle e_m, Te_n\rangle \langle e_n, f\rangle e_m =$$

$$\sum_{n=1}^{\infty} \langle e_n, f\rangle \Big(\sum_{m=1}^{\infty} \langle e_m, Te_n\rangle e_m\Big) = \sum_{n=1}^{\infty} \langle e_n, f\rangle \; Te_n = T\Big(\sum_{n=1}^{\infty} \langle e_n, f\rangle e_n\Big) = Tf$$

where at the second to last equality we used that the operator T is continuous.

Thus, $L_k f = Tf$ for every $f \in L^2(\mathbb{R}^d)$.

□

Exercise 14, Question 3. Question 1 shows that the operator $S : L^2(\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{L}^2(L^2(\mathbb{R}^d))$ given by $S(k) := L_k$ is an isometry and Question 2 shows that S is surjective. Therefore, the proof is complete.

$\square$

Exercise 14, Question 4. We present two approaches which are non-rigorous proofs of the desired equality.

Approach 1: By definition

$$\mathrm{Tr}(T) = \sum_{n=1}^{\infty} \Big\langle e_n, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} = \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \overline{e_n(x)}\ Te_n(x)\ dx =$$

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \overline{e_n(x)} k(x,y) e_n(y)\ dydx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x,y) \sum_{n=1}^{\infty} \overline{e_n(x)} e_n(y)\ dydx =$$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x,y) \delta(x-y)\ dydx = \int_{\mathbb{R}^d} k(x,x)\ dx.$$

Here we used that the kernel of the Identity operator $\mathrm{Id} = \sum_n |e_n\rangle\langle e_n|$ is the distribution $k(x,y) = \delta(x-y)$.

Approach 2: From expression (2) we have the following

$$\int_{\mathbb{R}^d} k(x,x)\ dx = \int_{\mathbb{R}^d} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big\langle e_m, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} e_{mn}(x,x)\ dx =$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big\langle e_m, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} e_m(x) \overline{e_n}(x)\ dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big\langle e_m, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} \Big\langle e_n, e_m \Big\rangle_{L^2(\mathbb{R}^d)} =$$

$$\sum_{n=1}^{\infty} \Big\langle e_n, Te_n \Big\rangle_{L^2(\mathbb{R}^d)} = \mathrm{Tr}(T).$$

The assumption that k is continuous is to avoid the difficulty of defining the $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ function k on the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ which is of measure 0.

$\square$

Exercise 14, Question 5. Since $\rho \in \mathcal{L}^1(L^2(\mathbb{R}^d))$ we have that ρ is a Hilbert-Schmidt operator in $L^2(\mathbb{R}^d)$ and from Question 2 we know that it has a kernel function which we denote by $\rho(x,v)$, $x, v \in \mathbb{R}^d$ (i.e. $\rho = L_\rho$ with a small abuse of notation). From the definition of the partial trace we know that ρ is the unique operator in $\mathcal{L}^1(L^2(\mathbb{R}^d))$ that satisfies

$$\mathrm{Tr}(k\rho) = \mathrm{Tr}((k \otimes \mathrm{Id})R) \tag{5}$$

for all $k \in \mathrm{Com}(L^2(\mathbb{R}^d))$. Since finite rank operators are dense in $(\mathrm{Com}(L^2(\mathbb{R}^d)), \|\cdot\|_{\mathrm{op}})$ and since finite rank operators are Hilbert-Schmidt operators (every Hilbert-Schmidt operator is also a compact operator) we obtain that the set $\mathcal{L}^2(L^2(\mathbb{R}^d))$ is dense in the Banach space $(\mathrm{Com}(L^2(\mathbb{R}^d)), \|\cdot\|_{\mathrm{op}})$. Hence, in (5) it suffices to consider $k \in \mathcal{L}^2(L^2(\mathbb{R}^d))$. But then with the use of Question 2 and Question 4 we may rewrite (5) as (again with a slight abuse of notation we will use $k(x,z)$ to denote the kernel function of the operator k)

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,z)\rho(z,x)\ dxdz = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,z)\Big[\int_{\mathbb{R}^m} R((z,y),(x,y))\ dy\Big]\ dxdz \tag{6}$$

or equivalently,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,z)\Big[\rho(z,x) - \int_{\mathbb{R}^m} R((z,y),(x,y))\ dy\Big]\ dxdz = 0 \tag{7}$$

for all $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Therefore, we must have

$$\rho(z,x) = \int_{\mathbb{R}^m} R((z,y),(x,y))\ dy$$

which is the desired equality.

$\square$