Exercise 15. For $\phi = \{\phi_k\}, \psi = \{\psi_k\}_k \in D(N)$ we trivially have
\[
\langle \phi, N\psi \rangle = \sum_{k=0}^{\infty} k \langle \phi_k, \psi_k \rangle h^k = \langle N\phi, \psi \rangle
\]
which means that $N$ is symmetric. It is straightforward to check that $\text{Ran}(N \pm i) = \mathcal{F}(h)$ since for $\phi \in \mathcal{F}(h)$ if we set
\[
\psi_k = \frac{1}{k+i} \phi_k, \text{ for } k \in \mathbb{N} \cup \{0\}
\]
we obtain $\psi = \{\psi_k\}_k \in D(N)$ with the property $(N + i)\psi = \phi$. Similarly, for the case of $-i$. Hence, by the basic criterion of self-adjointness we obtain that $N$ is self-adjoint.

For the operators $N|_{\mathcal{F}_a(h)}$ and $N|_{\mathcal{F}_s(h)}$ it suffices to observe that $N$ leaves the spaces $\mathcal{F}_a(h)$ and $\mathcal{F}_s(h)$ invariant (and so does for their orthogonal complements) which implies
\[
\text{Ran}(N|_{\mathcal{F}_a(h)} \pm i) = \mathcal{F}_a(h) \text{ and } \text{Ran}(N|_{\mathcal{F}_s(h)} \pm i) = \mathcal{F}_s(h).
\]
Again, by the basic criterion of self-adjointness we are done. \hfill \Box

Exercise 16. We consider scalars $\psi_0^0, \phi_0^0 \in \mathbb{C}$ and vectors $\psi_m^k, \phi_m^k \in h$. By denseness it suffices to prove the required equality for the vectors
\[
\phi = (\phi_0^0, \phi_1^0 \otimes \phi_2^0, \ldots, \phi_1^0 \otimes \ldots \otimes \phi_k^0, \ldots) \in \mathcal{F}(h)
\]
and
\[
\psi = (\psi_0^0, \psi_1^0 \otimes \psi_2^0, \ldots, \psi_1^0 \otimes \ldots \otimes \psi_k^0, \ldots) \in \mathcal{F}(h).
\]
By definition we have
\[
a(f)\psi = \left( \langle f, \psi_1^0 \rangle, 2^\frac{3}{2} \langle f, \psi_1^1 \rangle \psi_2^0, \ldots, k^\frac{1}{4} \langle f, \psi_1^k \rangle \psi_2^0 \otimes \ldots \otimes \psi_k^0, \ldots \right)
\]
and
\[
a^*(f)\phi = \left( 0, \phi_0^0 f, 2^\frac{3}{2} f \otimes \phi_1^0, 3^\frac{1}{4} f \otimes \phi_2^0 \otimes \phi_2^0, \ldots, (k + 1)^\frac{1}{2} f \otimes \phi_k^0 \otimes \ldots \phi_k^0, \ldots \right).
\]
Therefore, from the definition of the inner product in the Fock space $\mathcal{F}(h)$ we obtain
\[
\langle \phi, a(f)\psi \rangle = \phi_0^0 \langle f, \psi_1^0 \rangle + \langle f, \psi_1^1 \rangle \langle \phi_1^0, \psi_2^0 \rangle + \ldots + \langle f, \psi_1^{k+1} \rangle \langle \phi_1^k, \psi_2^{k+1} \rangle \langle \phi_k^k, \psi_2^{k+1} \rangle + \ldots
\]
and similarly
\[
\langle a^*(f)\phi, \psi \rangle = \phi_0^0 \langle f, \psi_1^0 \rangle + \langle f, \psi_1^1 \rangle \langle \phi_1^0, \psi_2^0 \rangle + \ldots + \langle f, \psi_1^{k+1} \rangle \langle \phi_1^k, \psi_2^{k+1} \rangle \langle \phi_k^k, \psi_2^{k+1} \rangle + \ldots
\]
which finishes the proof.

Observe that the previous argument works in the case of $\mathcal{F}_s(h)$ and $\mathcal{F}_a(h)$ as well. \hfill \Box
Exercise 17. Let us denote by $P_{a,N}$ the operator $P_a$ acting on functions on $\mathbb{R}^{3N}$. By denseness we may check the desired equality for functions of the form $f_1 \otimes \ldots \otimes f_n$ where $f_i \in h$ for all $i \in \{1, \ldots, n\}$. By definition we have

$$a^*_a(f)a_a(f) = P_{a,N+2}a^*(f)P_{a,N+1}a^*(f)P_{a,N} = P_{a,N+2}a^*(f)P_{a,N+1}a^*(f)P_{a,N}. $$

Then,

$$P_{a,N+1}a^*(f)P_{a,N}(f_1 \otimes \ldots \otimes f_n) = P_{a,N+1}\left(f \otimes P_{a,N}(f_1 \otimes \ldots \otimes f_n)\right) = P_{a,N+1}(f \otimes f_1 \otimes \ldots \otimes f_n)$$

since $P_{a,N+1}P_{a,N} = P_{a,N+1}$. Hence,

$$P_{a,N+2}a^*(f)P_{a,N+1}(f \otimes f_1 \otimes \ldots \otimes f_n) = P_{a,N+2}\left(f \otimes P_{a,N+1}(f \otimes f_1 \otimes \ldots \otimes f_n)\right) =$$

$$P_{a,N+2}(f \otimes f_1 \otimes \ldots \otimes f_n) = 0$$

where we used again that $P_{a,N+2}P_{a,N+1} = P_{a,N+2}$. The expression on the previous line is equal to zero since the first two functions of the tensor product are equal which means that the operator $P_a$ cancels them out when it acts on $f \otimes f_1 \otimes \ldots \otimes f_n$.

\[ \square \]

Exercise 18. By definition

$$a_a(f)\psi = P_a a(f)P_a \psi = P_a a(f)\psi$$

since $\psi \in L^2_a(\mathbb{R}^{3N})$. Hence,

$$(a_a(f)\psi)(x_1, \ldots, x_{N-1}) = P_a\left(N^{\frac{1}{2}} \int_{\mathbb{R}^3} f(y) \psi(y, x_1, \ldots, x_{N-1}) \, dy\right) =$$

$$(-1)^{N-1}N^{\frac{1}{2}} \int_{\mathbb{R}^3} \bar{f(x_N)} \psi(x_1, \ldots, x_N) \, dx_N$$

since again $\psi \in L^2_a(\mathbb{R}^{3N})$ and the signature (parity) of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & N-1 & N \\ N & 1 & 2 & \ldots & N-2 & N-1 \end{pmatrix} = (1 \, N \, N-1)(N-1 \, N-2) \ldots (3 \, 2)$$

is equal to $-1$.

Similarly, we have

$$a^*_a(f)\psi = P_a a(f)P_a \psi = P_a a(f)\psi$$

which implies

$$(a^*_a(f)\psi)(x_1, \ldots, x_{N+1}) = (N+1)^{\frac{1}{2}}P_a\left(f(x_1)\psi(x_2, \ldots, x_{N+1})\right) =$$

$$\frac{(N+1)^{\frac{1}{2}}}{(N+1)!} \sum_{\sigma \in S_{N+1}} (-1)^{\sigma} f(x_{\sigma(1)}) \psi(x_{\sigma(2)}, \ldots, x_{\sigma(N+1)}) =$$

$$\frac{(N+1)^{\frac{1}{2}}}{(N+1)!} N! \sum_{j=1}^{N+1} (-1)^{j+1} f(x_j) \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{N+1}) =$$
\[(N + 1)^{-\frac{1}{2}} \sum_{j=1}^{N+1} (-1)^{j+1} f(x_j) \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{N+1}) \]

since \( \psi \in L^2_a(\mathbb{R}^{3N}) \) and the summands which contain the expression
\[
\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{N+1})
\]
are \( N! \). We also used that the permutation that takes a summand of the second line and maps it to a summand of the third line has parity \((-1)^{j+1}\).

\[\Box\]

**Exercise 19.** From the previous Exercise we have (we drop the subindex \( a \) from the creation and annihilation operators)

1. \[(a(f)a(g)\psi)(x_1, \ldots, x_{N-2}) = \]
\[\left(-1\right)^{N-1}(-1)^{N-2}N^{\frac{1}{2}}(N-1)^{\frac{1}{2}} \int_{\mathbb{R}^3} f(x_N)g(x_{N-1})\psi(x_1, \ldots, x_N)dx_{N-1}dx_N = \]
\[-N^{\frac{1}{2}}(N-1)^{\frac{1}{2}} \int_{\mathbb{R}^3} f(x_N)g(x_{N-1})\psi(x_1, \ldots, x_N)dx_{N-1}dx_N = \]
\[N^{\frac{1}{2}}(N-1)^{\frac{1}{2}} \int_{\mathbb{R}^3} g(x_N)f(x_{N-1})\psi(x_1, \ldots, x_N)dx_{N-1}dx_N = -(a(g)a(f)\psi)(x_1, \ldots, x_{N-2})\]

where we used that since \( \psi \in L^2_a(\mathbb{R}^{3N}) \) we have
\[
\psi(x_1, \ldots, x_{N-2}, x_{N-1}, x_N) = -\psi(x_1, \ldots, x_{N-2}, x_N, x_{N-1}).
\]

Thus, \( \{a(f), a(g)\} = 0 \).

Similarly, we calculate

2. \[(a^*(f)a^*(g)\psi)(x_1, \ldots, x_{N+2}) = \]
\[(N + 2)^{-\frac{1}{2}}(N + 1)^{-\frac{1}{2}} \sum_{i=1}^{N+2} \sum_{j=1,j \neq i}^{N+2} (-1)^i(-1)^j f(x_i)g(x_j)\psi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{N+2}) = \]
\[-(a^*(g)a^*(f)\psi)(x_1, \ldots, x_{N+2})\]

since we have to exchange the variables \( x_i \leftrightarrow x_j \) and \( \psi \in L^2_a(\mathbb{R}^{3N}) \). Therefore, we obtain \( \{a^*(f), a^*(g)\} = 0 \).

Finally, for the last canonical anti-commutation relation we calculate

3. \[(a(f)a^*(g)\psi)(x_1, \ldots, x_N) = \]
\[(-1)^N \int_{\mathbb{R}^3} f(x_{N+1}) \sum_{j=1}^{N+1} (-1)^{j+1} g(x_j)\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{N+1})dx_{N+1}\]

and

4. \[(a^*(g)a(f)\psi)(x_1, \ldots, x_N) = \]
\[(-1)^{N-1} \sum_{j=1}^{N} (-1)^{j+1} g(x_j) \int_{\mathbb{R}^3} f(x_{N+1})\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{N+1})dx_{N+1}.\]
Therefore,
\[
\left( (a(f)a^*(g) + a^*(g)a(f))\psi \right)(x_1, \ldots, x_N) = \int_{\mathbb{R}^3} \overline{f(x_{N+1})}g(x_{N+1}) dx_{N+1} \psi(x_1, \ldots, x_N) =
\left\langle f, g \right\rangle_{L^2(\mathbb{R}^3)} \psi(x_1, \ldots, x_N)
\]
and the proof is complete. \qed