Exercise 15. For $\phi = \{\phi_k\}, \psi = \{\psi_k\}_k \in D(N)$ we trivially have

$$\langle \phi, N\psi \rangle = \sum_{k=0}^{\infty} k \langle \phi_k, \psi_k \rangle_{h^{\otimes k}} = \langle N\phi, \psi \rangle$$

which means that N is symmetric. It is straightforward to check that $\mathrm{Ran}(N \pm i) = \mathcal{F}(h)$ since for $\phi \in \mathcal{F}(h)$ if we set

$$\psi_k = \frac{1}{k+i}\ \phi_k, \text{ for } k \in \mathbb{N} \cup \{0\}$$

we obtain $\psi = \{\psi_k\}_k \in D(N)$ with the property $(N+i)\psi = \phi$. Similarly, for the case of $-i$. Hence, by the basic criterion of self-adjointness we obtain that N is self-adjoint.

For the operators $N|_{\mathcal{F}_a(h)}$ and $N|_{\mathcal{F}_s(h)}$ it suffices to observe that N leaves the spaces $\mathcal{F}_a(h)$ and $\mathcal{F}_s(h)$ invariant (and so does for their orthogonal complements) which implies

$$\mathrm{Ran}(N|_{\mathcal{F}_a(h)} \pm i) = \mathcal{F}_a(h) \text{ and } \mathrm{Ran}(N|_{\mathcal{F}_s(h)} \pm i) = \mathcal{F}_s(h).$$

Again, by the basic criterion of self-adjointness we are done.

□

Exercise 16. We consider scalars $\psi_0^0, \phi_0^0 \in \mathbb{C}$ and vectors $\psi_m^k, \phi_m^k \in h$. By denseness it suffices to prove the required equality for the vectors

$$\phi = (\phi_0^0, \phi_1^1, \phi_1^2 \otimes \phi_2^2, \ldots, \phi_1^k \otimes \ldots \otimes \phi_k^k, \ldots) \in \mathcal{F}(h)$$

and

$$\psi = (\psi_0^0, \psi_1^1, \psi_1^2 \otimes \psi_2^2, \ldots, \psi_1^k \otimes \ldots \otimes \psi_k^k, \ldots) \in \mathcal{F}(h).$$

By definition we have

$$a(f)\psi = \Big(\langle f, \psi_1^1 \rangle, 2^{\frac{1}{2}} \langle f, \psi_1^2 \rangle \psi_2^2, \ldots, k^{\frac{1}{2}} \langle f, \psi_1^k \rangle \psi_2^k \otimes \ldots \otimes \psi_k^k, \ldots \Big)$$

and

$$a^*(f)\phi = \Big(0, \phi_0^0 f, 2^{\frac{1}{2}} f \otimes \phi_1^1, 3^{\frac{1}{2}} f \otimes \phi_1^2 \otimes \phi_2^2, \ldots, (k+1)^{\frac{1}{2}} f \otimes \phi_1^k \otimes \ldots \otimes \phi_k^k, \ldots \Big).$$

Therefore, from the definition of the inner product in the Fock space $\mathcal{F}(h)$ we obtain

$$\Big\langle \phi, a(f)\psi \Big\rangle = \phi_0^0 \langle f, \psi_1^1 \rangle + \langle f, \psi_1^2 \rangle \langle \phi_1^1, \psi_2^2 \rangle + \ldots + \langle f, \psi_1^{k+1} \rangle \langle \phi_1^k, \psi_2^{k+1} \rangle \ldots \langle \phi_k^k, \psi_{k+1}^{k+1} \rangle + \ldots$$

and similarly

$$\Big\langle a^*(f)\phi, \psi \Big\rangle = \phi_0^0 \langle f, \psi_1^1 \rangle + \langle f, \psi_1^2 \rangle \langle \phi_1^1, \psi_2^2 \rangle + \ldots + \langle f, \psi_1^{k+1} \rangle \langle \phi_1^k, \psi_2^{k+1} \rangle \ldots \langle \phi_k^k, \psi_{k+1}^{k+1} \rangle + \ldots$$

which finishes the proof.

Observe that the previous argument works in the case of $\mathcal{F}_s(h)$ and $\mathcal{F}_a(h)$ as well.

□

Exercise 17. Let us denote by $P_{a,N}$ the operator P_a acting on functions on $\mathbb{R}^{3N}$. By denseness we may check the desired equality for functions of the form $f_1 \otimes \ldots \otimes f_n$ where $f_i \in h$ for all $i \in \{1, \ldots, n\}$. By definition we have

$$a_a^*(f)a_a^*(f) = P_{a,N+2}a^*(f)P_{a,N+1}P_{a,N+1}a^*(f)P_{a,N} = P_{a,N+2}a^*(f)P_{a,N+1}a^*(f)P_{a,N}.$$

Then,

$$P_{a,N+1}a^*(f)P_{a,N}(f_1 \otimes \ldots \otimes f_n) = P_{a,N+1}\Big(f \otimes P_{a,N}(f_1 \otimes \ldots \otimes f_n)\Big) = P_{a,N+1}(f \otimes f_1 \otimes \ldots \otimes f_n)$$

since $P_{a,N+1}P_{a,N} = P_{a,N+1}$. Hence,

$$P_{a,N+2}a^*(f)P_{a,N+1}(f \otimes f_1 \otimes \ldots \otimes f_n) = P_{a,N+2}\Big(f \otimes P_{a,N+1}(f \otimes f_1 \otimes \ldots \otimes f_n)\Big) =$$

$$P_{a,N+2}(f \otimes f \otimes f_1 \otimes \ldots \otimes f_n) = 0$$

where we used again that $P_{a,N+2}P_{a,N+1} = P_{a,N+2}$. The expression on the previous line is equal to zero since the first two functions of the tensor product are equal which means that the operator P_a cancels them out when it acts on $f \otimes f \otimes f_1 \otimes \ldots \otimes f_n$.

□

Exercise 18. By definition

$$a_a(f)\psi = P_a a(f) P_a \psi = P_a a(f)\psi$$

since $\psi \in L_a^2(\mathbb{R}^{3N})$. Hence,

$$(a_a(f)\psi)(x_1, \ldots, x_{N-1}) = P_a\Big(N^{\frac{1}{2}} \int_{\mathbb{R}^3} \overline{f(y)}\ \psi(y, x_1, \ldots, x_{N-1})\ dy\Big) =$$

$$(-1)^{N-1}N^{\frac{1}{2}} \int_{\mathbb{R}^3} \overline{f(x_N)}\ \psi(x_1, \ldots, x_N)\ dx_N$$

since again $\psi \in L_a^2(\mathbb{R}^{3N})$ and the signature (parity) of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & N-1 & N \\ N & 1 & 2 & \cdots & N-2 & N-1 \end{pmatrix} = (1\ N)(N\ N-1)(N-1\ N-2)\cdots(3\ 2)$$

is equal to $N-1$.

Similarly, we have

$$a_a^*(f)\psi = P_a a(f) P_a \psi = P_a a(f)\psi$$

which implies

$$(a_a^*(f)\psi)(x_1, \ldots, x_{N+1}) = (N+1)^{\frac{1}{2}} P_a\Big(f(x_1)\psi(x_2, \ldots, x_{N+1})\Big) =$$

$$\frac{(N+1)^{\frac{1}{2}}}{(N+1)!} \sum_{\sigma \in S_{N+1}} (-1)^{\sigma}\ f(x_{\sigma(1)})\ \psi(x_{\sigma(2)}, \ldots, x_{\sigma(N+1)}) =$$

$$\frac{(N+1)^{\frac{1}{2}}}{(N+1)!}\ N! \sum_{j=1}^{N+1} (-1)^{j+1}\ f(x_j)\ \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{N+1}) =$$

$$(N+1)^{-\frac{1}{2}} \sum_{j=1}^{N+1} (-1)^{j+1}\ f(x_j)\ \psi(x_1,\ldots,\hat{x}_j,\ldots,x_{N+1})$$

since $\psi \in L^2_a(\mathbb{R}^{3N})$ and the summands which contain the expression

$$\psi(x_1,\ldots,\hat{x}_j,\ldots,x_{N+1})$$

are $N!$. We also used that the permutation that takes a summand of the second line and maps it to a summand of the third line has parity $(-1)^{j-\sigma+1}$.

□

Exercise 19. From the previous Exercise we have (we drop the subindex a from the creation and annihilation operators)

(1)
$$(a(f)a(g)\psi)(x_1 \ldots, x_{N-2}) =$$
$$(-1)^{N-1}(-1)^{N-2}N^{\frac{1}{2}}(N-1)^{\frac{1}{2}} \int_{\mathbb{R}^3} \overline{f(x_N)g(x_{N-1})}\psi(x_1,\ldots,x_N)dx_{N-1}dx_N =$$
$$-N^{\frac{1}{2}}(N-1)^{\frac{1}{2}} \int_{\mathbb{R}^3} \overline{f(x_N)g(x_{N-1})}\psi(x_1,\ldots,x_N)dx_{N-1}dx_N =$$
$$N^{\frac{1}{2}}(N-1)^{\frac{1}{2}} \int_{\mathbb{R}^3} \overline{g(x_N)f(x_{N-1})}\psi(x_1,\ldots,x_N)dx_{N-1}dx_N = -(a(g)a(f)\psi)(x_1,\ldots,x_{N-2})$$

where we used that since $\psi \in L^2_a(\mathbb{R}^{3N})$ we have

$$\psi(x_1,\ldots,x_{N-2},x_{N-1},x_N) = -\psi(x_1,\ldots,x_{N-2},x_N,x_{N-1}).$$

Thus, $\{a(f),a(g)\} = 0$.

Similarly, we calculate

(2)
$$(a^*(f)a^*(g)\psi)(x_1,\ldots,x_{N+2}) =$$
$$(N+2)^{-\frac{1}{2}}(N+1)^{-\frac{1}{2}} \sum_{i=1}^{N+2} \sum_{j=1,j\neq i}^{N+2} (-1)^i(-1)^j f(x_i)g(x_j)\psi(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j\ldots,x_{N+2}) =$$
$$-(a^*(g)a^*(f)\psi)(x_1,\ldots,x_{N+2})$$

since we have to exchange the variables $x_i \leftrightarrow x_j$ and $\psi \in L^2_a(\mathbb{R}^{3N})$. Therefore, we obtain $\{a^*(f),a^*(g)\} = 0$.

Finally, for the last canonical anti-commutation relation we calculate

(3)
$$(a(f)a^*(g)\psi)(x_1,\ldots,x_N) =$$
$$(-1)^N \int_{\mathbb{R}^3} \overline{f(x_{N+1})} \sum_{j=1}^{N+1} (-1)^{j+1} g(x_j)\psi(x_1,\ldots,\hat{x}_j,\ldots,x_{N+1})dx_{N+1}$$

and

(4)
$$(a^*(g)a(f)\psi)(x_1,\ldots,x_N) =$$
$$(-1)^{N-1} \sum_{j=1}^{N} (-1)^{j+1} g(x_j) \int_{\mathbb{R}^3} \overline{f(x_{N+1})}\psi(x_1,\ldots,\hat{x}_j,\ldots,x_{N+1})dx_{N+1}.$$

Therefore,

$$\Big((a(f)a^*(g)+a^*(g)a(f))\psi\Big)(x_1,\dots,x_N)=\int_{\mathbb{R}^3}\overline{f(x_{N+1})}g(x_{N+1})dx_{N+1}\psi(x_1,\dots,x_N)=$$

$$\Big\langle f,g\Big\rangle_{L^2(\mathbb{R}^3)}\psi(x_1,\dots,x_N)$$

and the proof is complete.

□