Exercise 20. We start with the operators $a_s(f), a_s^*(f)$ on $\mathcal{F}_s(L^2(\mathbb{R}^3))$ for an $f \in L^2(\mathbb{R}^3)$. Since $f^{\otimes N} \in L_s^2(\mathbb{R}^{3N})$ we have

$$a_s^*(f)f^{\otimes N} = \sqrt{N+1}\ f^{\otimes(N+1)} \text{ and } a_s(f)f^{\otimes N} = \sqrt{N}\ \|f\|_2^2\ f^{\otimes(N-1)}$$

which implies that

$$\frac{\|a_s^*(f)f^{\otimes N}\|_{L^2(\mathbb{R}^{3(N+1)})}}{\|f^{\otimes N}\|_{L^2(\mathbb{R}^{3N})}} = \sqrt{N+1}\ \|f\|_2 \to \infty \text{ as } N \to \infty$$

and similarly

$$\frac{\|a_s(f)f^{\otimes N}\|_{L^2(\mathbb{R}^{3(N-1)})}}{\|f^{\otimes N}\|_{L^2(\mathbb{R}^{3N})}} = \sqrt{N}\ \|f\|_2 \to \infty \text{ as } N \to \infty.$$

Hence, the operators $a_s(f), a_s^*(f)$ are not bounded on $\mathcal{F}_s(L^2(\mathbb{R}^3))$.

We continue with the operators $a_a(f), a_a^*(f)$ on $\mathcal{F}_a(L^2(\mathbb{R}^3))$ for an $f \in L^2(\mathbb{R}^3)$. Notice that the example we used before, namely functions of the form $f^{\otimes N}$ do not lie in $L_a^2(\mathbb{R}^{3N})$. We will show that

$$\|a_a(f)\psi\|_{L^2(\mathbb{R}^{3(N-1)})} \leq \|f\|_2\ \|\psi\|_{L^2(\mathbb{R}^{3N})} \tag{1}$$

for every $N \in \mathbb{N}$ and $\psi \in P_a(L^2(\mathbb{R}^{3N}) = L_a^2(\mathbb{R}^{3N})$. Indeed, observe that (we drop the subindex a and use Exercise 17 i.e. $a^*(f)a^*(f) = 0$ and Exercise 19)

$$(a^*(f)a(f))^2 = a^*(f)a(f)a^*(f)a(f) = a^*(f)a(f)a^*(f)a(f) + a^*(f)a^*(f)a(f)a(f) = \tag{2}$$

$$a^*(f)\{a(f), a^*(f)\}a(f) = a^*(f)a(f)\|f\|_2^2.$$

Now we may derive that

$$\|a^*(f)a(f)\psi\|^2 = \langle a^*(f)a(f)\psi, a^*(f)a(f)\psi\rangle = \langle \psi, (a^*(f)a(f))^2\psi\rangle = \tag{3}$$

$$\|f\|_2^2\langle \psi, a^*(f)a(f)\psi\rangle \leq \|f\|_2^2\|\psi\|\|a^*(f)a(f)\psi\| \Rightarrow$$

$$\|a^*(f)a(f)\psi\| \leq \|f\|_2^2\|\psi\|. \tag{4}$$

Finally, we have

$$\|a(f)\psi\|^2 = \langle \psi, a^*(f)a(f)\psi\rangle \leq \|\psi\|\|a^*(f)a(f)\psi\| \leq \|f\|_2^2\|\psi\|^2 \tag{5}$$

which implies that $\|a(f)\| \leq \|f\|_2$ or in other words the operator a_a is bounded on $\mathcal{F}_a(L^2(\mathbb{R}^3))$. Since

$$\|a(f)\| = \sup_{\phi,\psi \in \mathcal{F}_a(L^2(\mathbb{R}^3))} \left|\left\langle \phi, a(f)\psi \right\rangle\right| = \sup_{\phi,\psi \in \mathcal{F}_a(L^2(\mathbb{R}^3))} \left|\left\langle a^*(f)\phi, \psi \right\rangle\right| = \|a^*(f)\|$$

we obtain the boundedness of a_a^* too and the proof is complete.

□

Exercise 21. We start with the direction $\Rightarrow$. Suppose that q is a closed semi-bounded quadratic form with form domain $Q(q) \subset \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space (with norm $\|\cdot\|$). By assumption there exists $M > 0$ such that

$$q(\psi, \psi) \geq -M\|\psi\|^2 \tag{6}$$

for all $\psi \in Q(q)$. In addition, assume that $\{\phi_n\}_n$ is a sequence in $Q(q)$ and $\phi \in \mathcal{H}$ such that

$$q(\phi_n - \phi_m, \phi_n - \phi_m) \to 0 \text{ in } \mathbb{C} \text{ and } \phi_n \to \phi \text{ in } \mathcal{H} \tag{7}$$

as $n, m \to \infty$. Since q is closed we know that the pair $(Q(q), \|\cdot\|_q)$ is a Banach space where

$$\|\phi\|_q = \sqrt{q(\phi, \phi) + (M+1)\|\phi\|^2} \tag{8}$$

for $\phi \in Q(q)$. Then from (7) and (8) we have that the sequence $\{\phi_n\}_n$ is Cauchy in $(Q(q), \|\cdot\|_q)$ which means that there is $\phi_0 \in Q(q)$ such that $\phi_n \to \phi_0$ in $(Q(q), \|\cdot\|_q)$ as $n \to \infty$. But

$$\|\phi_n - \phi_0\|_q^2 = q(\phi_n - \phi_0, \phi_n - \phi_0) + (M+1)\|\phi_n - \phi_0\|^2 \geq \|\phi_n - \phi_0\|^2 \geq 0 \tag{9}$$

where we used (6). Because the LHS of (9) goes to 0 as $n \to \infty$ we get that $\|\phi_n - \phi_0\| \to 0$ as $n \to \infty$ or in other words $\phi = \phi_0 \in Q(q)$. Finally observe that

$$q(\phi_n - \phi, \phi_n - \phi) = \|\phi_n - \phi\|_q^2 - (M+1)\|\phi_n - \phi\|^2 \to 0 \text{ as } n \to \infty$$

and the claim is proved.

We continue with the direction $\Leftarrow$. We have to show that the pair $(Q(q), \|\cdot\|_q)$ is a Banach space. Hence, let $\{\phi_n\}_n$ be a Cauchy sequence in $(Q(q), \|\cdot\|_q)$ i.e.

$$\|\phi_n - \phi_m\|_q^2 = q(\phi_n - \phi_m, \phi_n - \phi_m) + (M+1)\|\phi_n - \phi_m\|^2 \to 0 \text{ as } n \to \infty. \tag{10}$$

As we did in (9) using the semi-boundedness of q we obtain that $\|\phi_n - \phi_m\| \to 0$ which means that the sequence $\{\phi_n\}_n$ is Cauchy is the Hilbert space $\mathcal{H}$. Thus, there is $\phi \in \mathcal{H}$ with the property $\phi_n \to \phi$ as $n \to \infty$ in $\mathcal{H}$. But then we use our assumption to derive that $\phi \in Q(q)$ and that $q(\phi_n - \phi, \phi_n - \phi) \to 0$ in $\mathbb{C}$. In other words, we found $\phi \in Q(q)$ such that

$$\|\phi_n - \phi\|_q^2 = q(\phi_n - \phi, \phi_n - \phi) + (M+1)\|\phi_n - \phi\|^2 \to 0 \text{ as } n \to \infty.$$

Thus, the pair $(Q(q), \|\cdot\|_q)$ is a complete metric space and the proof is complete.

Next we want to show that the quadratic form $q : C_c(\mathbb{R}) \times C_c(\mathbb{R}) \to \mathbb{C}$ in $L^2(\mathbb{R})$ given by $q(f, g) = \overline{f(0)}g(0)$ is not closed. Fix a $\psi \in C_c(\mathbb{R})$ such that $\psi(0) = 1$ and define the sequence $\phi_n(x) = \psi(nx) \in C_c(\mathbb{R})$. We claim that for $\phi = 0 \in C_c(\mathbb{R})$ we have

$$\phi_n \to \phi \text{ in } L^2(\mathbb{R}) \text{ and } q(\phi_n - \phi_m, \phi_n - \phi_m) \to 0 \text{ in } \mathbb{C} \text{ as } n, m \to \infty$$

but

$$\lim_{n\to\infty} q(\phi_n - \phi, \phi_n - \phi) \neq 0.$$

Indeed,

$$\|\phi_n\|_2^2 = \int_{\mathbb{R}} |\phi_n(x)|^2 \, dx = \int_{\mathbb{R}} |\psi(nx)|^2 \, dx = \frac{1}{n}\int_{\mathbb{R}} |\psi(y)|^2 \, dy = \frac{1}{n}\|\psi\|_2^2 \to 0$$

as $n \to \infty$. Also, by the definition of q we have

$$q(\phi_n - \phi_m, \phi_n - \phi_m) = |\phi_n(0)|^2 + |\phi_m(0)|^2 - 2\mathrm{Re}(\overline{\phi_n(0)}\phi_m(0)) = 2|\psi(0)|^2 - 2|\psi(0)|^2 = 0$$

and

$$q(\phi_n - \phi, \phi_n - \phi) = q(\phi_n, \phi_n) = \overline{\phi_n(0)}\phi_n(0) = |\phi_n(0)|^2 = |\psi(0)|^2 = 1.$$

Therefore, by the criterion we proved before we obtain that the quadratic form q is not closed. Notice that the same proof shows that q can not have a closed extension $\tilde{q}$ since this would imply that $\psi(0) = 0$ which is not true by the initial choice of the function ψ.

□

Exercise 22. The domain of the operator $-\Delta - \frac{1}{|x|} = \frac{5}{4}$ is $H^2(\mathbb{R}^3)$ and the form domain of the quadratic form $q(\phi, \psi) + \frac{5}{4}\langle \phi, \psi \rangle$ is $H^1(\mathbb{R}^3)$. Trivially, $H^2(\mathbb{R}^3) \subset H^1(\mathbb{R}^3)$ and we have shown in class that $q(\phi, \phi) + \frac{5}{4}\langle \phi, \phi \rangle \geq \|\phi\|_2^2$ for all $\phi \in H^1(\mathbb{R}^3)$. Therefore, it remains to show that for $\phi \in H^2(\mathbb{R}^3)$ and $\psi \in H^1(\mathbb{R}^3)$ we have the equality

$$\Big\langle \Big(-\Delta - \frac{1}{|x|} + \frac{5}{4} \Big) \phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)} = q(\phi, \psi) + \frac{5}{4} \Big\langle \phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)}.$$

Using integration by parts (and approximation by smooth functions) we observe that the LHS equals

$$\Big\langle -\Delta\phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)} + \Big\langle -\frac{1}{|x|}\phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)} + \frac{5}{4}\Big\langle \phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)} =$$

$$\Big\langle \nabla\phi, \nabla\psi \Big\rangle_{L^2(\mathbb{R}^3)} + \Big\langle -\frac{1}{|x|}\phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)} + \frac{5}{4}\Big\langle \phi, \psi \Big\rangle_{L^2(\mathbb{R}^3)} = q(\phi, \psi) + \frac{5}{4}\langle \phi, \psi \rangle$$

which proves the claim.

□