Exercise 23. Since the quadratic form q_H is positive, by Theorem 15.4 (see HW sheet 7 Exercise 21) it suffices to consider a sequence $\{\phi_n\}_n \subset H^{1,1}(\mathbb{R})$ and a $\phi \in L^2(\mathbb{R})$ such that

$$q_H(\phi_n - \phi_m, \phi_n - \phi_m) \to 0 \text{ in } \mathbb{C} \text{ and } \phi_n \to \phi \text{ in } L^2(\mathbb{R}) \text{ as } n, m \to \infty. \tag{1}$$

If we show that $\phi \in H^{1,1}(\mathbb{R})$ and $q(\phi_n - \phi, \phi_n - \phi) \to 0$ as $n \to \infty$ we are done. By (1) and the definition of the $H^{1,1}(\mathbb{R})$ norm it is trivial to see that the sequence $\{\phi_n\}_n$ is Cauchy in the Hilbert space $(H^{1,1}(\mathbb{R}), \|\cdot\|_{H^{1,1}})$. Hence, there is a function $\phi_0 \in H^{1,1}(\mathbb{R})$ with the property that

$$\phi_n \to \phi_0 \text{ in } H^{1,1}(\mathbb{R}) \text{ as } n \to \infty. \tag{2}$$

But since convergene in the $H^{1,1}(\mathbb{R})$ norm implies convergence in the $L^2(\mathbb{R})$ norm we have that in $L^2(\mathbb{R})$, $\phi_0 = \lim_{n\to\infty} \phi_n = \phi$. Thus, $\phi = \phi_0 \in H^{1,1}(\mathbb{R})$. In addition, by (2), the fact that $\phi_n \to \phi$ in $L^2(\mathbb{R})$ and the definition of the $H^{1,1}(\mathbb{R})$ norm we obtain that

$$q_H(\phi_n - \phi, \phi_n - \phi) \to 0$$

as $n \to \infty$ and the proof is complete.

□

Exercise 24. Since the quadratic form q is closable we know that the completion of the space $(Q(q), \|\cdot\|_q)$ is a closed subspace of the Hilbert space $\mathcal{H}$. By the definition of the closure $\bar{q}$ of q we have that the normed space $(Q(\bar{q}), \|\cdot\|_{\bar{q}})$ is the completion of the normed space $(Q(q), \|\cdot\|_q)$, that $Q(q)$ is a form core of the quadratic form $\bar{q}$ and that actually for every $x \in Q(\bar{q})$ we have

$$\bar{q}(x, x) = \lim_{n\to\infty} q(x_n, x_n) \tag{3}$$

where $\{x_n\}_n \subset Q(q)$ such that $x_n \to x$ in the $\|\cdot\|_{\bar{q}}$ norm (this can be easily checked to be well defined, i.e. if $\{y_n\}_n \subset Q(q)$ such that $y_n \to x$ in $\|\cdot\|_{\bar{q}}$ then $\lim_n q(y_n, y_n) = \lim_n q(x_n, x_n)$).

By assumption there is $M > 0$ such that

$$q(x, x) \geq -M\|x\|^2 \tag{4}$$

for all $x \in Q(q)$. We have to prove that

$$\bar{q}(x, x) \geq -M\|x\|^2 \tag{5}$$

for all $x \in Q(\bar{q})$. Hence, let $x \in Q(\bar{q})$. Since $Q(q)$ is a dense subset of the normed space $(Q(\bar{q}), \|\cdot\|_{\bar{q}})$ we know that there is a sequence $\{x_n\}_n$ in $Q(q)$ such that $x_n \to x$ in $\|\cdot\|_{\bar{q}}$. But then we have trivially

$$\bar{q}(x, x) = \lim_{n\to\infty} q(x_n, x_n) \geq -M \lim_{n\to\infty} \|x_n\|^2 = -M\|x\|^2$$

since convergence in $\|\cdot\|_{\bar{q}}$ implies convergence in the norm $\|\cdot\|$ of the Hilbert space $\mathcal{H}$.

□

Exercise 25. We will identify the completion of the space $(C_c(\mathbb{R}), \|\cdot\|_q)$ as the Hilbert space $\mathbb{C} \times L^2(\mathbb{R})$ equipped with the norm

$$\|(z, g)\| := \left(|z|^2 + \|g\|_2^2\right)^{\frac{1}{2}} \text{ for } z \in \mathbb{C} \text{ and } g \in L^2(\mathbb{R}). \tag{6}$$

We define the linear map $F : C_c(\mathbb{R}) \to \mathbb{C} \times L^2(\mathbb{R})$ by

$$F(f) = (f(0), f) \in \mathbb{C} \times L^2(\mathbb{R}) \text{ for } f \in C_c(\mathbb{R}). \tag{7}$$

It is straightforward to see that F is an isometry between the normed spaces $(C_c(\mathbb{R}), \|\cdot\|_q)$ and $(\mathrm{Ran}(F), \|\cdot\|)$. Hence, the completion of $(C_c(\mathbb{R}), \|\cdot\|_q)$ is isometric to the completion of $(\mathrm{Ran}(F), \|\cdot\|)$ in the Hilbert space $(\mathbb{C} \times L^2(\mathbb{R}), \|\cdot\|)$. If we show that the set $\mathrm{Ran}(F)$ is dense in $\mathbb{C} \times L^2(\mathbb{R})$ then we will obtain that the completion of $(C_c(\mathbb{R}), \|\cdot\|_q)$ is isometric to the space $(\mathbb{C} \times L^2(\mathbb{R}), \|\cdot\|)$ and the proof will be complete.

Thus, consider $(z, f) \in \mathbb{C} \times L^2(\mathbb{R})$. By the denseness of $C_c(\mathbb{R})$ in $L^2(\mathbb{R})$ we can find a sequence $\{f_n\}_n \subset C_c(\mathbb{R})$ with the property that

$$f_n \to f \text{ in } L^2(\mathbb{R}). \tag{8}$$

For every $n \in \mathbb{N}$ we can find a function $g_n \in C_c(\mathbb{R})$ such that

$$g_n(0) = z - f_n(0) \text{ and } \|g_n\|_2 \leq \frac{1}{n}. \tag{9}$$

Then we have that the sequence $\{f_n + g_n\}_n \subset C_c(\mathbb{R})$ and has the properties $f_n(0) + g_n(0) = z$ and

$$\|f_n + g_n - f\|_2 \leq \|f_n - f\|_2 + \|g_n\|_2 \leq \|f_n - f\|_2 + \frac{1}{n} \to 0 \text{ as } n \to \infty$$

or in other words $f_n + g_n \to f$ in $L^2(\mathbb{R})$. Therefore, the sequence $(f_n(0) + g_n(0), f_n + g_n) \in \mathrm{Ran}(F)$ and converges to the pair (z, f) in $\mathbb{C} \times L^2(\mathbb{R})$ which means we are done.

□

Exercise 26, Question 1. Trivially we have

$$\psi_1(x) = a^* \psi_0(x) = \frac{\sqrt{2}}{\pi^{\frac{1}{4}}} \, x e^{-\frac{x^2}{2}}$$

which implies by integration by parts that

$$\|\psi_1\|_2^2 = 1.$$

Now assume that for some $n \in \mathbb{N}$ we have $\|(a^*)^{(n)} \psi_0\|_2 = \sqrt{n!}$. We will show that $\|(a^*)^{(n+1)} \psi_0\|_2 = \sqrt{(n+1)!}$. Indeed,

$$\|(a^*)^{(n+1)} \psi_0\|_2^2 = \|a^* \psi_n\|_2^2 = \langle a^* \psi_n, a^* \psi_n \rangle = \langle \psi_n, a a^* \psi_n \rangle = \langle \psi_n, (a^* a + \mathrm{Id}) \psi_n \rangle =$$
$$\langle \psi_n, a^* a \psi_n \rangle + \langle \psi_n, \psi_n \rangle = \langle \psi_n, n \psi_n \rangle + n! = n n! + n! = (n+1)!$$

and the proof is complete by induction.

□

Exercise 26, Question 2. Assume that ψ is an eigenfunction of the operator $N = a^* a$ and that for some $n \in \mathbb{N}$ and $C \in \mathbb{C}$ we have

$$a^{(n)} \psi = C \psi_0 = D e^{-\frac{x^2}{2}}. \tag{10}$$

Then

$$\left(\frac{d}{dx} + x \right)^{(n)} \psi = D e^{-\frac{x^2}{2}} \tag{11}$$

or equivalently,

$$\left(\frac{d}{dx}\right)^{(n)}\left(e^{\frac{x^2}{2}}\psi\right) = D$$

which implies that

$$e^{\frac{x^2}{2}}\psi = \frac{D}{n!}x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0.$$

Hence, ψ belongs to the space (notice that the space of solutions of the previously mentioned ODE is an affine subspace of M_n)

$$M_n = \left\{p(x)e^{-\frac{x^2}{2}} \;\middle|\; p(x) \in \mathbb{C}[x], \text{degree}(p(x)) \leq n\right\}. \tag{12}$$

Observe that the span$\{\psi_0, \psi_1, \ldots, \psi_n\}$ is a subset of M_n (this is because from the definition of the ψ_j's we know that, up to constants, $\psi_j = (x - d/dx)^j e^{-\frac{x^2}{2}}$) and actually we have $\dim M_n = n + 1 = \dim(\text{span}\{\psi_0, \psi_1, \ldots, \psi_n\})$. Therefore,

$$M_n = \text{span}\{\psi_0, \psi_1, \ldots, \psi_n\}$$

which implies that $\psi = \psi_0$ or $\psi = \psi_1$ or $\ldots$ or $\psi = \psi_n$ (up to constants). This is the case since ψ is an eigenfunction of the operator N and the $\psi_0, \psi_1, \ldots, \psi_n$ are also eigenfunctions of the operator N corresponding to different eigenvalues. But since $a^{(n)}\psi_0 = a^{(n)}\psi_1 = \ldots = a^{(n)}\psi_{n-1} = 0$ we must have that $\psi = \psi_n$ (up to constants again) which completes the proof.

□