Exercise 27. Suppose that $\phi \in L^2(\mathbb{R}^{3(n+1)})$ and $\psi \in L^2(\mathbb{R}^{3n})$. Then by definition we have

$$\Big\langle \hat{\phi}, a^*(k)\hat{\psi} \Big\rangle = \Big\langle a(k)\hat{\phi}, \hat{\psi} \Big\rangle = \overline{\Big\langle \hat{\psi}, a(k)\hat{\phi} \Big\rangle} = C\overline{\Big\langle \psi, \int e^{-ikx} a(x)dx\ \phi \Big\rangle} =$$

$$D\Big\langle \int e^{-ikx} a(x)dx\ \phi, \psi \Big\rangle = D\Big\langle \phi, \int e^{ikx} a^*(x)dx\ \psi \Big\rangle$$

which implies the first equality we want to show, i.e.

$$\mathcal{F}^{-1} a^*(k) \mathcal{F} = D \int e^{ikx} a^*(x)dx.$$

For the second equality let us consider $\phi, \psi \in L^2(\mathbb{R}^{3n})$. Then again by definition and Plancherel's identity we have

$$\Big\langle \hat{\phi}, \int a^*(k)a(k)dk\ \hat{\psi} \Big\rangle = n \int \overline{\hat{\phi}(k_1, \ldots, k_{n-1}, k)}\hat{\psi}(k_1, \ldots, k_{n-1}, k) \prod_{j=1}^{n-1} dk_k dk =$$

$$n \int \overline{\phi(x_1, \ldots, x_{n-1}, x)}\psi(x_1, \ldots, x_{n-1}, x) \prod_{j=1}^{n-1} dx_j dx = \Big\langle \phi, \int a^*(x)a(x)dx\ \psi \Big\rangle$$

which means

$$\mathcal{F}^{-1} \int a^*(k)a(k)\mathcal{F} = \int a^*(x)a(x)dx.$$

□

Exercise 28. Observe that

$$\Big\langle \phi, a(f_{x,1,\infty})\phi \Big\rangle = \Big\langle \phi, a(i\mathrm{div}_x g_{x,1})\phi \Big\rangle = -i\Big\langle \phi, a\Big(\sum_{j=1}^{3} \partial_j g_{x,1}^{(j)}\Big)\phi \Big\rangle = \tag{1}$$

$$-i\sum_{j=1}^{3} \Big\langle \phi, a(\partial_j g_{x,1}^{(j)})\phi \Big\rangle.$$

But for an $\phi \in C_c^\infty(\mathbb{R}^3) \otimes L_s^2(\mathbb{R}^{3(n-1)})$ and a $\psi \in C_c^\infty(\mathbb{R}^3) \otimes L_s^2(\mathbb{R}^{3n})$ we have by definition and integration by parts that

$$\Big\langle \phi, a(\partial_j g_{x,1}^{(j)})\psi \Big\rangle = \int \overline{\phi(x, k_1, \ldots, k_{n-1})} \int \overline{\partial_j g_{x,1}^{(j)}(k_n)}\psi(x, k_1, \ldots, k_n)dk_n \prod_{m=1}^{n-1} dk_m = \tag{2}$$

$$-\Big\langle \phi, a(g_{x,1}^{(j)})\partial_j \psi \Big\rangle - \Big\langle \partial_j \phi, a(g_{x,1}^{(j)})\psi \Big\rangle$$

since the derivative falls one time on the function ψ and one time on the function ϕ. Inserting this into the last expression in the chain of equalities of (1) we obtain that

$$\langle \phi, a(f_{x,1,\infty})\phi \rangle = i\sum_{j=1}^{3} \Big[\langle \partial_{x_j}\phi, a(g_{x,1}^{(j)})\phi \rangle + \langle a^*(g_{x,1}^{(j)})\phi, \partial_{x_j}\phi \rangle \Big]$$

and the proof is complete.

□

Exercise 29. As we did in class we write the quadratic form q_H as

$$q_H(\phi,\psi) = q_{H_0}(\phi,\psi) + \beta(\phi,\psi) + \left\langle \phi, \sum_{1\le i<j\le n} \frac{U}{|x_i - x_j|}\psi \right\rangle \tag{3}$$

where

$$q_{H_0}(\phi,\psi) = \sum_{j=1}^{n} \left\langle \nabla_{x_j}\phi, \nabla_{x_j}\psi \right\rangle + \left\langle N^{\frac{1}{2}}\phi, N^{\frac{1}{2}}\psi \right\rangle \tag{4}$$

and

$$\beta(\phi,\psi) = \sum_{j=1}^{n} \left(\left\langle \phi, a(f_{x_j})\psi \right\rangle + \left\langle a(f_{x_j})\phi, \psi \right\rangle \right). \tag{5}$$

We treated the term $\beta(\phi,\psi)$ in class. Also, the term $\langle \phi, \sum_{1\le i<j\le n} \frac{U}{|x_i-x_j|}\psi\rangle$ can be treated using the KLMN Theorem (or the Kato-Rellich Theorem) in exactly the same way as in HW sheet 4 Exercise 13 of the previous semester. For the term q_{H_0} it is obvious to see that it is a positive (and hence, semi-bounded) quadratic form. We claim that it is also closable. To see this, we have to identify the completion of its form domain

$$C_c^\infty(\mathbb{R}^{3n}) \otimes \mathcal{F}_{s,c}(L^2(\mathbb{R}^3))$$

with the norm

$$\|\phi\|_{q_{H_0}} = \sqrt{\sum_{j=1}^{n} \|\nabla_{x_j}\phi\|_{\mathcal{H}}^2 + \|N^{\frac{1}{2}}\phi\|_{\mathcal{H}}^2 + \|\phi\|_{\mathcal{H}}^2}$$

as a subset of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3n}) \otimes \mathcal{F}_s(L^2(\mathbb{R}^3))$. If we have a Cauchy sequence $\{\phi_n\}$ in $C_c^\infty(\mathbb{R}^{3n}) \otimes \mathcal{F}_{s,c}(L^2(\mathbb{R}^3))$ with respect to the norm $\|\cdot\|_{q_{H_0}}$ then $\{\phi_n\}$ and $\{\nabla_{x_j}\phi_n\}$ is obviously Cauchy in the Hilbert space $\mathcal{H}$ and in addition, $\{\phi_n\}$ is Cauchy in the space $(D(N^{\frac{1}{2}}), \|N^{\frac{1}{2}}\cdot\|_{\mathcal{H}})$. But the operator $N^{\frac{1}{2}}$ is self-adjoint (and so closed) and we know from last semester that this last space is complete. Hence,

$$\operatorname{compl}(C_c^\infty(\mathbb{R}^{3n}) \otimes \mathcal{F}_{s,c}(L^2(\mathbb{R}^3)), \|\cdot\|_{q_{H_0}}) \subset \left\{ \phi \in \mathcal{H} \;\middle|\; \nabla_x\phi, N^{\frac{1}{2}}\phi \in \mathcal{H} \right\} \subset \mathcal{H}.$$

and so the quadratic form q_{H_0} is closable.

□