Exercise 27. Suppose that $\phi \in L^2(\mathbb{R}^{3(n+1)})$ and $\psi \in L^2(\mathbb{R}^{3n})$. Then by definition we have

$$\left\langle \hat{\phi}, a^*(k) \hat{\psi} \right\rangle = \left\langle a(k) \hat{\phi}, \psi \right\rangle = \left\langle \psi, a(k) \hat{\phi} \right\rangle = C \left\langle \psi, \frac{1}{i} \int e^{-ikx} a(x) dx \phi \right\rangle =$$

$$D \left\langle \int e^{-ikx} a(x) dx \phi, \psi \right\rangle = D \left\langle \phi, \int e^{ikx} a^*(x) dx \psi \right\rangle$$

which implies the first equality we want to show, i.e.

$$\mathcal{F}^{-1} a^*(k) \mathcal{F} = D \int e^{ikx} a^*(x) dx.$$  

For the second equality let us consider $\phi, \psi \in L^2(\mathbb{R}^{3n})$. Then again by definition and Plancherel’s identity we have

$$\left\langle \hat{\phi}, \int a^*(k) a(k) dk \hat{\psi} \right\rangle = n \int \hat{\phi}(k_1, \ldots, k_{n-1}, k) \hat{\psi}(k_1, \ldots, k_{n-1}, k) \prod_{j=1}^{n-1} dk_j dk =$$

$$n \int \hat{\phi}(x_1, \ldots, x_{n-1}, x) \hat{\psi}(x_1, \ldots, x_{n-1}, x) \prod_{j=1}^{n-1} dx_j dx = \left\langle \phi, \int a^*(x) a(x) dx \psi \right\rangle$$

which means

$$\mathcal{F}^{-1} \int a^*(k) a(k) \mathcal{F} = \int a^*(x) a(x) dx.$$  

Exercise 28. Observe that

(1)  

$$\left\langle \phi, a(f_{x,1,\infty}) \phi \right\rangle = \left\langle \phi, a(i \text{div}_x g_{x,1}) \phi \right\rangle = -i \left\langle \phi, a \left( \sum_{j=1}^{3} \partial_j g_{x,1}^{(j)} \right) \phi \right\rangle =$$

$$-i \sum_{j=1}^{3} \left\langle \phi, a(\partial_j g_{x,1}^{(j)}) \phi \right\rangle.$$  

But for an $\phi \in C_c^\infty(\mathbb{R}^3) \otimes L^2_\ast(\mathbb{R}^{3(n-1)})$ and a $\psi \in C_c^\infty(\mathbb{R}^3) \otimes L^2_\ast(\mathbb{R}^{3n})$ we have by definition and integration by parts that

(2)  

$$\left\langle \phi, a(\partial_j g_{x,1}^{(j)}) \psi \right\rangle = \int \phi(x, k_1, \ldots, k_{n-1}) \partial_j \int g_{x,1}^{(j)}(k_n) \psi(x, k_1, \ldots, k_n) dk_n \prod_{m=1}^{n-1} dk_m =$$

$$- \left\langle \phi, a(g_{x,1}^{(j)}) \partial_j \psi \right\rangle - \left\langle \partial_j \phi, a(g_{x,1}^{(j)}) \psi \right\rangle$$

since the derivative falls one time on the function $\psi$ and one time on the function $\phi$. Inserting this into the last expression in the chain of equalities of (1) we obtain that

$$\left\langle \phi, a(f_{x,1,\infty}) \phi \right\rangle = i \sum_{j=1}^{3} \left[ \left( \partial_{x,j} \phi, a(g_{x,1}^{(j)}) \phi \right) + \left( a^*(g_{x,1}^{(j)}) \phi, \partial_{x,j} \phi \right) \right]$$

and the proof is complete.
Exercise 29. As we did in class we write the quadratic form \( q_H \) as

\[
q_H(\phi, \psi) = q_{H_0}(\phi, \psi) + \beta(\phi, \psi) + \left( \phi, \sum_{1 \leq i < j \leq n} \frac{U}{|x_i - x_j|} \psi \right)
\]

where

\[
q_{H_0}(\phi, \psi) = \sum_{j=1}^{n} \left( \nabla x_j \phi, \nabla x_j \psi \right) + \left( N^{\frac{1}{2}} \phi, N^{\frac{1}{2}} \psi \right)
\]

and

\[
\beta(\phi, \psi) = \sum_{j=1}^{n} \left( \left( \phi, a(f_{x_j}) \psi \right) + \left( a(f_{x_j}) \phi, \psi \right) \right).
\]

We treated the term \( \beta(\phi, \psi) \) in class. Also, the term \( \langle \phi, \sum_{1 \leq i < j \leq n} \frac{U}{|x_i - x_j|} \psi \rangle \) can be treated using the KLMN Theorem (or the Kato-Rellich Theorem) in exactly the same way as in HW sheet 4 Exercise 13 of the previous semester. For the term \( q_{H_0} \) it is obvious to see that it is a positive (and hence, semi-bounded) quadratic form. We claim that it is also closable. To see this, we have to identify the completion of its form domain

\[
C^\infty_c(\mathbb{R}^{3n}) \otimes F_{s,c}(L^2(\mathbb{R}^{3}))
\]

with the norm

\[
\| \phi \|_{q_{H_0}} = \sqrt{\sum_{j=1}^{n} \| \nabla x_j \phi \|_{H}^2 + \| N^{\frac{1}{2}} \phi \|_{H}^2 + \| \phi \|_{H}^2}
\]

as a subset of the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^{3n}) \otimes F_{s,c}(L^2(\mathbb{R}^{3})) \). If we have a Cauchy sequence \( \{ \phi_n \} \) in \( C^\infty_c(\mathbb{R}^{3n}) \otimes F_{s,c}(L^2(\mathbb{R}^{3})) \) with respect to the norm \( \| \cdot \|_{q_{H_0}} \) then \( \{ \phi_n \} \) and \( \{ \nabla x_j \phi_n \} \) is obviously Cauchy in the Hilbert space \( \mathcal{H} \) and in addition, \( \{ \phi_n \} \) is Cauchy in the space \( (D(N^{\frac{1}{2}}), \| \cdot \|_{\mathcal{H}}) \). But the operator \( N^{\frac{1}{2}} \) is self-adjoint (and so closed) and we know from last semester that this last space is complete. Hence,

\[
\text{compl}(C^\infty_c(\mathbb{R}^{3n}) \otimes F_{s,c}(L^2(\mathbb{R}^{3})), \| \cdot \|_{q_{H_0}}) \subset \left\{ \phi \in \mathcal{H} \mid \nabla x \phi, N^{\frac{1}{2}} \phi \in \mathcal{H} \right\} \subset \mathcal{H}.
\]

and so the quadratic form \( q_{H_0} \) is closable. 

\[\square\]