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Mathematical Quantum Mechanics 2
(aka. Mathematical Methods in Quantum Mechanics 2)
7. Homework

Exercise 16

Let V be self-adjoint on G and compact. Consider the Lieb-Thirring inequality

$$\operatorname{tr}_{L^2(\mathbb{R}^d)\otimes G}\big((P^2\otimes \mathbf{1}+V)_-^\gamma\big) \leqslant C^{op}_{\gamma,d}L^{cl}_{\gamma,d}\int_{\mathbb{R}^d}\operatorname{tr}_G\big(V_-^{\gamma+\frac{d}{2}}\big)\,dx, \tag{1}$$

with optimal constant $C^{op}_{\gamma,d}L^{cl}_{\gamma,d}$, where $L^{cl}_{\gamma,d}$ is the (fixed) classical Lieb-Thirring constant. Assume the following three properties:

$$C^{op}_{\gamma,d} \leqslant C^{op}_{\gamma,n}C^{op}_{\gamma+\frac{n}{2},d-n} \quad \forall 1\leqslant n\leqslant d-1, \qquad \text{(Submultiplicativity)} \tag{2}$$

$$1 \leqslant C^{op}_{\gamma_1,d} \leqslant C^{op}_{\gamma_2,d} \quad \forall \gamma_1 \geqslant \gamma_2, \qquad \text{(Monotonicity)} \tag{3}$$

$$C^{op}_{\frac{3}{2},1} = 1, \qquad \text{(Exact value).} \tag{4}$$

Show that

$$C^{op}_{0,d} \leqslant C^{op}_{0,n} \quad \forall d > n \geqslant 3.$$

Exercise 17

Given V such that $(P^2+V)_- \in \mathcal{S}^1(\mathcal{H})$, and $\sigma_{ess}(P^2+V)\subset[0,\infty)$, show that

$$-\operatorname{tr}\big((P^2+V)_-\big) = \inf_{\sigma\in\mathcal{S}^1, 0\leqslant\sigma\leqslant 1}\operatorname{tr}(\sigma(P^2+V)), \tag{5}$$

where $0\leqslant\sigma\leqslant 1$ is understood in the sense that $\sigma=\sum_{j\in\mathbb{N}}\lambda_j\,|\varphi_j\rangle\,\langle\varphi_j|$ for some orthonormal family $(\varphi_j)_j$ in $\mathcal{H}$, and $0\leqslant\lambda_j\leqslant 1$, $\sum_{j=1}^\infty \lambda_j<\infty$.

Exercise 18

Consider a non-negative, convex function $G:\mathbb{R}_+\to\mathbb{R}_+$ and define its Legendre transformation as

$$G(s) = \sup_{t\geqslant 0}(st - F(t)).$$

Prove that the following are equivalent:

- The Lieb-Thirring bound

$$\operatorname{tr}\big((P^2+V)_-\big) \leqslant \int_{\mathbb{R}^d} G(V_-(x))\,dx, \tag{6}$$

- The Thomas-Fermi bound

$$\langle \psi, \sum_{n=1}^{N} P^2 \psi \rangle_{\Lambda^N L^2(\mathbb{R}^d)} \geqslant \int_{\mathbb{R}^d} F(\varrho_\psi(x))\, dx, \tag{7}$$

for all $\psi \in \Lambda^N L^2(\mathbb{R}^d)$ with norm one, and where

$$\varrho_\psi(x) = N \int_{\mathbb{R}^{(N-1)d}} |\psi(x, x_2, \ldots, x_N)|^2\, dx_2 \ldots dx_N.$$

— bitte wenden —