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Mathematical Quantum Mechanics 2
(aka. Mathematical Methods in Quantum Mechanics 2)
1. Homework

Exercise 1

The reason why we know that $ab \leq \frac{1}{2}(a^2 + b^2)$ for any $a, b \in \mathbb{R}$ is that $(a - b)^2 \geq 0$, since the square of a real number is always non–negative.
Exactly the same reasoning is used in the proof of the Cauchy–Schwarz inequality and we will use this idea once more for the proof of Hardy's inequality

$$\langle \nabla\psi, \nabla\psi \rangle \geq \frac{(d-2)^2}{4} \langle \psi, \frac{1}{|x|^2}\psi \rangle \,.$$

for all $d \geq 3$ and all $\psi \in \mathcal{D}(P)$, the domain of the momentum operator (which is the Sobolev space $H^1 = \{\psi \in L^2 : \eta\widehat{\psi} \in L^2\}$ where $\widehat{\psi}$ is the Fourier transform of ψ).

a) For a vector field $W : \mathbb{R}^d \to \mathbb{R}^d$, which is differentiable and locally bounded, except maybe at zero, show that

$$\|(\nabla + W)\psi\|^2 = \langle \nabla\psi, \nabla\psi \rangle + \langle \psi, (W^2 - \mathrm{div} W)\psi \rangle$$

for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus 0)$, i.e., ψ is infinitely often differentiable and has compact support $K = \mathrm{supp}(\psi)$ and $0 \notin K$.

b) Now play with different choices for the vector field W to derive Hardy's inequality for functions $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus 0)$

(Hint: You want to try something of the form $W(x) = \alpha x/|x|^\beta$ for suitable $\alpha, \beta \in \mathbb{R}$)

c) You might know that $\mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{0\})$ is dense in $\mathcal{D}(P)$ in dimension $d \geq 3$. That is, for any $\psi \in \mathcal{D}(P)$, there exist a sequence $\psi_n \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{0\})$ such that both $\|\psi - \psi_n\|$ and $\|P(\psi - \psi_n)\|$ converge to zero as $n \to \infty$ (when $d \geq 3$).

Use this to finish the proof of Hardy's inequality for $d \geq 3$.

Exercise 2

For the Schrödinger operator

$$H = P^2 - \frac{1}{|x|} = -\Delta - \frac{1}{|x|} \,, \; x \in \mathbb{R}^3$$

of a Hydrogen atom the essential spectrum (for physicists, the set of energies of states which move to infinity) coincides with the half-line $[0, +\infty)$. According to the minimax principle an operator A with essential spectrum on the positive half-line has infinite number of negative eigenvalues if one can find an infinite sequence of trial functions $\{\psi_n\}, n \in \mathbb{N}$ with disjoint supports, such that for each of $\{\psi_n\}$ we have $(A\psi_n, \psi_n) < 0$.
The goal of this exercise is to show that Hydrogen has infintly many bound states *without solving the eigenvalue equation explicitly*!

a) Prove that the Hydrogen atom has at least one negative eigenvalue.

(Hint: Take any function ψ which is infinitely often differentiable with $\psi(x) = 0$ if $|x| \leq 1$ or if $|x| \geq 2$. The consider the scaled function ψ_R given by $\psi_R(x) = \psi(x/R)$. Show that one has $\langle \psi_R, H\psi_R \rangle < 0$ for large enough R)

b) Use the above ideas to find a sequence ψ_n of functions such that ψ_n have pairwise disjoint supports ($\mathrm{supp}(\psi_n) \cap \mathrm{supp}(\psi_m) = \emptyset$ when $n \neq m$) and $\langle \psi_n, H\psi_n \rangle < 0$ for all $n \in \mathbb{N}$.

c) Explain why this argument does not work for a Schrödinger operator $P^2 + V$ with potential $V = (1 + |x|)^{-3}$.