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WS 2022/23
28.11.2022

Mathematical Quantum Mechanics 2
(aka. Mathematical Methods in Quantum Mechanics 2)
4. Homework

Exercise 7

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. For $\varphi_1 \in \mathcal{H}_1, \varphi_2 \in \mathcal{H}_2$ we denote by $\varphi_1 \otimes \varphi_2$ the bilinear map, which maps from $\mathcal{H}_1 \times \mathcal{H}_2$ to $\mathbb{C}$, given by

$$(\varphi_1 \otimes \varphi_2)\langle \psi_1, \psi_2 \rangle := (\psi_1, \varphi_1)_{\mathcal{H}_1} (\psi_2, \varphi_2)_{\mathcal{H}_2}. \tag{1}$$

Let Σ be the set of all finite combinations of such bilinear forms. We define an inner product $(\cdot, \cdot)$ on Σ by

$$(\varphi \otimes \psi, \eta \otimes \mu) := (\varphi, \eta)_{\mathcal{H}_1} (\psi, \mu)_{\mathcal{H}_2} \tag{2}$$

and extend this to Σ by linearity. Show that $(\cdot, \cdot)$ is well defined and positive definite.

Definition: We define $\mathcal{H}_1 \otimes \mathcal{H}_2$ to be the completion under the inner product $(\cdot, \cdot)$ defined above. $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called the *tensor product* of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Note that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is gain a Hilbert space. What is its inner product, i.e., scalar product?

Exercise 8

Let $\mathcal{H}$ be an Hilbert space. A bounded operator $A \in \mathcal{L}(H)$ is called trace class operator if

$$\|A\|_1 := \operatorname{tr}(|A|) < \infty.$$

The set of all trace class operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{S}_1(\mathcal{H})$. The set of all compact operators on $\mathcal{H}$ is denoted by $\operatorname{Com}(\mathcal{H})$.

The goal of this exercise is to show that $\mathcal{S}_1(\mathcal{H})$ is isomorphic to $\mathcal{K}(\mathcal{H})^*$. Thus $\mathcal{S}_1(\mathcal{H})$ is – or can be identified with the help of the usual isomorphism game – the *dual space of the compact operators* on the Hilbert space. The map which Φ does this identification is given by

$$\Phi : \mathcal{S}_1(\mathcal{H}) \to \operatorname{Com}(\mathcal{H})^*, A \mapsto \operatorname{tr}(A\cdot)\,,\ \text{i.e., } \Phi(A)[C] = \operatorname{tr}(AC)$$

Let $F \in \mathcal{K}(H)^*$, we need to find an $A = A_F \in \mathcal{S}_1(\mathcal{H})$ such that $F(C) = \operatorname{tr}(A_F C)$ for all compact operators C. For this we will use that the finite rank operators are dense in the compact operators with respect to the operator norm. The simplest finite rank operators, are the rank one operators which, in Dirac notation, are given by by $|\varphi\rangle\langle\psi|$. So we start with them.

- **a)** Show there exists a unique $A = A_F \in \mathcal{B}(H)$ with $\langle \psi, A\varphi \rangle = F(|\varphi\rangle\langle\psi|)$ for any $\psi, \varphi \in H$, where the map $|\varphi\rangle\langle\psi|$ is defined by $|\varphi\rangle\langle\psi| f = \langle \psi, f \rangle \varphi$.

  Hint: Can you define the bounded operator A via its "matrix elements" $\langle \psi, A\varphi \rangle$?
- **b)** Prove that $A \in \mathcal{S}_1(\mathcal{H})$ and $\|A\|_1 \leqslant \|F\|_{\operatorname{Com}(\mathcal{H})^*}$.
- **c)** Prove that $\Phi(A) = F$.
- **d)** Prove that $\|A\|_1 \geqslant \|F\|_{\operatorname{Com}(\mathcal{H})^*}$.

Use the steps above to conclude the statement.

— bitte wenden —

Exercise 9

Let $\Gamma \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Show there exists a unique trace class operator $\gamma = \gamma_\Gamma \in \mathcal{S}_1(\mathcal{H}_1)$ such that

$$\operatorname{tr}((B \otimes \mathbf{1}_{\mathcal{H}_2})\Gamma) = \operatorname{tr}(B\gamma) \tag{3}$$

for any $B \in \mathcal{L}(\mathcal{H}_1)$. Here $\mathbf{1}_{\mathcal{H}_2}$ denotes the identity on $\mathcal{H}_2$, $\operatorname{tr}((B \otimes \mathbf{1}_{\mathcal{H}_2})\Gamma)$ is the trace over the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, and $\operatorname{tr}(B\gamma)$ is the trace over $\mathcal{H}_1$.
In math γ is called *partial trace* of Γ over $\mathcal{H}_2$. In physics lingo it is the *reduced density matrix* of Γ.

Hints:

a) First prove prove the existence of a partial trace when you additionally assume that $B \in \operatorname{Com}(\mathcal{H}_1)$. Have a sharp look at the map $B \mapsto \operatorname{tr}((B \otimes \mathbf{1}_{\mathcal{H}_2})\Gamma)$.

You might want to use the previous exercise....

b) Use part a) to show the existence of a partial trace for all $B \in \mathcal{L}(\mathcal{H}_1)$.