lecture 2. Mathematical Physics SS 2013

I.2.2 Operators on Hilbert spaces

A linear operator (or short operator) on a Hilbert space $\mathcal{H}$ is a tuple $(A, D(A))$,

$D(A)$ = domain of $A \subset \mathcal{H}$ linear & dense subset.

and $A : \mathcal{D}(A) \to \mathcal{H}$ s.t. $\forall\, u, v \in \mathcal{D}(A), \alpha, \beta \in \mathbb{C}$

$$A(\alpha u + \beta v) = \alpha A u + \beta A v$$

Ex · identity $\mathbb{1} = id : \psi \to \psi$ $\quad \mathcal{D}(\mathbb{1}) = \mathcal{H}$

· Multiplication by a coordinate

$$x_j : \psi \mapsto x_j \cdot \psi$$

i.e., $(x_j \psi)(x) := x_j \cdot \psi(x)$.

$$D(x_j) = \left\{ \psi \in L^2 \,/\, \int x_j^2 |\psi(x)|^2 dx < \infty \right\}$$

$$\underset{\neq}{\subset} L^2(\mathbb{R}^2).$$

Note $\mathcal{C}_0^\infty(\mathbb{R}^d)$ = set of ∞ often diff. fcns with compact support

is dense in $L^2(\mathbb{R}^d)$ and $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(x_j)$

so $\mathcal{D}(x_j)$ is dense in $L^2(\mathbb{R}^d)$.

- Momentum operator

$$P_j : \psi \mapsto -i\hbar \partial_j \psi$$

$$\mathcal{D}(P_j) = \{ \psi \in L^2 \mid \partial_j \psi \in L^2 \}$$
$$= \{ \psi \in L^2 \mid \int k_j^2 \, |\hat{\psi}(k)|^2 \, dk < \infty \}$$

again $\mathcal{C}_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(P_j)$, so $\mathcal{D}(P_j)$ is dense in L^2.

- Laplace $\quad \Delta : \psi \mapsto \sum_{j=1}^{d} \partial_j^2 \psi$

$$\mathcal{C}_c^\infty \subset \mathcal{D}(\Delta) = \{ \psi \in L^2 \mid \Delta\psi \in L^2 \}$$
$$= \{ \psi \in L^2 \mid \int k^2 \, |\hat{\psi}(k)|^2 \, dk < \infty \}$$

- Multiplication by a function $V : \mathbb{R}^d \to \mathbb{R}$.

$$V : \psi \mapsto V\psi \quad , \text{i.e. } (V\psi)(x) := V(x)\psi(x).$$

$$\mathcal{D}(V) = \{ \psi \in L^2 \mid \int V(x)^2 |\psi(x)|^2 \, dx < \infty \}$$

$\supset \mathcal{C}_c^\infty$ if $V \in L^2_{loc}$

- Schrödinger operator

$$H : \psi \mapsto -\frac{\hbar^2}{2m} \Delta\psi + V\psi$$

$$\mathcal{D}(H) = \mathcal{D}(H) \cap \mathcal{D}(V).$$

- Integral operator

$$\mathbb{K} : \psi \mapsto \int k(\cdot, y)\, \psi(y)\, dy$$

i.e., $(\mathbb{K}\psi)(x) := \int k(x, y)\, \psi(y)\, dy$

$k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is called <u>integral kernel</u> of $\mathbb{K}$

<u>Def 3</u> An operator A on $\mathcal{H}$ is bounded, if

$$\|A\| := \underbrace{\sup\left(\|A\psi\| \mid \psi \in \mathcal{D}(A),\ \|\psi\| = 1\right)}_{\text{operator norm}} < \infty$$

$$= \sup\left(\|A\psi\| \mid \psi \in \mathcal{D}(A),\ \|\psi\| \le 1\right)$$

$$= \sup\left(\|A\psi\| \mid \psi \in \mathcal{D}(A),\ \|\psi\| < 1\right)$$

$\mathcal{L}(\mathcal{H})\ (= \mathcal{B}(\mathcal{H}))$ = set of all bounded linear operators.

(Note $(\mathcal{L}(\mathcal{H}), \|.\|)$ is a Banach space)

<u>Lemma 4</u> If an operator A satisfies $\|A\psi\| \le C\|\psi\|$ for all ψ in a dense domain $D \subset \mathcal{H}$, then its extends uniquely to a bounded operator (also called A) on all of $\mathcal{H}$ with $\|A\psi\| \le C\|\psi\|$ $\forall \psi \in \mathcal{H}$.

Idea of proof: If $u \in \mathcal{H} \Rightarrow \exists (u_n)_n \subset D$, $u_n \to u$ (D is dense in $\mathcal{H}$). So, since

$$\|Au_n - Au_m\| \leq C\|u_n - u_m\| \to 0 \quad n,m \to \infty$$

$(Au_n)_n$ is Cauchy in $\mathcal{H}$. So

$$v := \lim_{n\to\infty} Au_n \quad \text{exists}$$

Define $Au := v = \lim_{n\to\infty} Au_n$. Done.

Remark: One needs to check consistency, i.e., if $u_n \to u$ and $\tilde{u}_n \to u$, then $v = \lim Au_n = \lim A\tilde{u}_n$ and that A is bounded with $\|A\| \leq C$.

Def 5: An operator A closed, if for any seq. $(u_n)_n \subset \mathcal{D}(A)$ with $u_n \to u$ and $Au_n \to v$ one has $u \in \mathcal{D}(A)$ and $Au = v$.

Another way of saying this is that the graph of A

$$G(A) := \{(u, Au) \mid u \in \mathcal{D}(A)\} \subset \mathcal{H} \times \mathcal{H}$$

is closed in $\mathcal{H} \times \mathcal{H}$.

- An operator A on $\mathcal{H}$ is <u>symmetric</u> if

$$\langle u, Av\rangle = \langle Au, v\rangle \quad \forall u, v \in \mathcal{D}(A)$$

<u>Theorem 6</u> A closed or symmetric operator on a Hilbert space $\mathcal{H}$ with $\mathcal{D}(A) = \mathcal{H}$ is bounded!

<u>Pf</u>: For closed operators this is the closed graph thm. For a symmetric operator this is the Hellinger-Toeplitz thm; see Reed-Simon I

Assume that A is symmetric & $\mathcal{D}(A) = \mathcal{H}$

Claim: A is a closed operator!

Indeed: let $(u_n)_n \in \mathcal{D}(A) = \mathcal{H}$, $u_n \to u$ and $Au_n \to v$. Need only show $Au = v$!

Let $w \in \mathcal{H}$. Then, since A is symmetric

$$\langle w, v\rangle = \lim_{n\to\infty} \underbrace{\langle w, Au_n\rangle}_{= \langle Aw, u_n\rangle}$$

$$= \langle Aw, u\rangle = \langle w, Au\rangle$$

$$\Rightarrow \quad \langle w, v - Au\rangle = 0$$

Choose $w = v - Au$

$$\Rightarrow \|v - Au\|^2 = \langle w, v - Au\rangle = 0.$$

so $v = Au$. $\boxtimes$

I.2.3 Inverses and their estimates

Given an operator A on $\mathcal{H}$, B is called the inverse of A if $\mathcal{D}(B) = \mathrm{Ran}(A)$, $\mathcal{D}(A) = \mathrm{Ran}(B)$ and

$$BA = \mathbb{1}|_{\mathrm{Ran}(B)} \quad \& \quad AB = \mathbb{1}|_{\mathrm{Ran}(A)}$$

where $\mathrm{Ran}(A) := \{Au \mid u \in \mathcal{D}(A)\} \subset \mathcal{H}$.

is the <u>range</u> of A

<u>Remark</u>: There is at most one inverse (uniqueness) we denote it by A^{-1}.

$$Au = f \iff A^{-1}f = u.$$

A convenient criterion for an operator to have an inverse is that it is one-to-one, i.e.,

$Au = 0 \Rightarrow u = 0$.

or that $\quad Ker(A) := \{ u \in \mathcal{D}(A) \mid Au = 0 \} = \{0\}$

i.e, the kernel of A is trivial.

An operator A is <u>invertible</u> if A has a <u>bounded inverse</u>. Since bounded ops. are def. on all of $\mathcal{H}$ we thus need that $A : \mathcal{D}(A) \to \mathcal{H}$ is one-to-one and onto, $Ran(A) = \mathcal{H}$.

<u>Remark</u>: If A is closed and one-to-one and onto then A has an inverse A^{-1}, then A^{-1} is also closed and $\mathcal{D}(A^{-1}) = \mathcal{H}$, so A^{-1} is automatically bounded.

<u>Lemma 7</u> a) If A and C are invertible operators on $\mathcal{H}$ and C is bounded, then CA is defined on $\mathcal{D}(CA) = \mathcal{D}(A)$ and invertible with

$$(CA)^{-1} = A^{-1} C^{-1}.$$

Pf: H W.

Thm 8. If the gen. A is invertible and B is bounded with $\underbrace{\|BA^{-1}\|}_{\text{operator norm}} < 1$, then $A+B$ defined on $\mathcal{D}(A+B) = \mathcal{D}(A)$ is invertible

Pf:

Write $A+B = (\underbrace{\mathbb{1} + BA^{-1}}_{=:C})A$

then $\|BA^{-1}\| < 1$ implies that C is invertible (see HW!), $A+B = (1+C)A$ is invertible by Lemma 7.

I.2.4 Self-adjointness

Recall that $A : \mathcal{D}(A) \to \mathcal{H}$ is symmetric, if

$$\langle u, Av \rangle = \langle Au, v \rangle \quad \forall u, v \in \mathcal{D}(A).$$

We want to replace this with the help of adjoints

Adjoint A^* of A should satisfy

$$\langle A^* \varphi, \psi \rangle = \langle \varphi, A\psi \rangle \qquad (1)$$

for all $\psi \in \mathcal{D}(A)$ $\varphi \in \mathcal{D}(A^*)$.

How to get $\mathcal{D}(A^*)$? Note that (1) says there is $w = A^* u$ s.t.

$$\langle \psi, A\varphi\rangle = \underbrace{\langle w, \varphi\rangle}_{\text{extends boundedly to all } \varphi \in \mathcal{H}!} \quad \forall \varphi \in \mathcal{D}(A)$$

<u>Def 9</u> (Adjoint) Given $A : \mathcal{D}(A) \to \mathcal{H}$, the adjoint A^* is defined as

$$\mathcal{D}(A^*) := \{\psi \in \mathcal{H} \mid \text{The map } \mathcal{D}(A) \ni \varphi \mapsto \langle \psi, A\varphi\rangle \text{ extend boundedly to all of } \mathcal{H}\}$$

$$= \{\psi \in \mathcal{H} \mid \exists\, C_\psi < \infty \text{ s.t. } |\langle \psi, A\varphi\rangle| \le C_\psi \cdot \|\varphi\| \; \forall \varphi \in \mathcal{D}(A)\}$$

In this case, $\varphi \mapsto \ell(\varphi) := \langle \psi, A\varphi\rangle$ is a bounded lin. functional (1st def. on $\mathcal{D}(A)$, then extended to all of $\mathcal{H}$ by continuity) and by Riesz rep. thm there exists $w \in \mathcal{H}$ s.t. $\ell(\varphi) = \langle w, \varphi\rangle$ $\forall v$.

Put

$$A^*\psi := w.$$

Then $\langle A^*\psi, \varphi\rangle = \langle \psi, A\varphi\rangle$

$\forall \psi \in \mathcal{D}(A^*), \varphi \in \mathcal{D}(A)$.

Remark Note that A symmetric says that

$$\langle A\psi, \varphi\rangle = \langle \psi, A\varphi\rangle \quad \forall\ u, v \in \mathcal{D}(A)$$

so if A is symmetric, then $\mathcal{D}(A) \subset \mathcal{D}(A^*)$

and $A^*|_{\mathcal{D}(A)} = A$, i.e., A^* extends A

We write this as $A \subset A^*$.

So A symmetric $\Longleftrightarrow$ A^* extends A (or $A \subset A^*$).

Def 10 An operator A on $\mathcal{H}$ is self-adjoint, if

$A^* = A$, i.e. A is symmetric and $\mathcal{D}(A^*) = \mathcal{D}(A)$.

Thm 11 (Fundamental criterion for self-adjointness)

The following are equivalent

a) A is self-adjoint

b) $\mathrm{Ran}\,(A \mp i) = \mathcal{H}$

c) $\mathrm{Ker}\,(A^* \mp i) = \{0\}$.

Here $A \mp i = A \mp i\mathbb{1}$

Pf: e.g. Reed-Simon I.