Lecture 3. Mathematical Physics SS 2013

Note Title 4/18/2013

Recall from last time

Def 10 An operator A on $\mathcal{H}$ is self-adjoint, if

$A^* = A$, i.e. A is symmetric and $\mathcal{D}(A^*) = \mathcal{D}(A)$.

Thm 11 (Fundamental criterion for self-adjointness)

The following are equivalent

a) A is self-adjoint

b) $\operatorname{Ran}(A \mp i) = \mathcal{H}$

c) $\operatorname{Ker}(A^* \mp i) = \{0\}$.

Here $A \mp i = A \mp i\mathbb{1}$

Pf: e.g. Reed-Simon I.

That we use $\pm i$ above is not a restriction:

Lemma 12 Let A be a symmetric operator. If

$\operatorname{Ran}(A - z_0) = \mathcal{H}$ for some $z_0 \in \mathbb{C}$, $\operatorname{Im} z_0 > 0$, then

$\operatorname{Ran}(A - z) = \mathcal{H}$ for all $z \in \mathbb{C}$, $\operatorname{Im} z > 0$

(Similar for z with $\operatorname{Im} z < 0$).

Moreover, if A is self-adjoint, then $A-z$ is invertible for every $z \in \mathbb{C} \setminus \mathbb{R}$ and

$$\| (A-z)^{-1} \| \leq \frac{1}{|\operatorname{Im} z|}$$

Remark: So b) in Thm 10 holds

$\iff$ for some $\mu > 0$ one has $\operatorname{Ran}(A \mp i\mu) = \mathcal{H}$.

proof of Lem 12: $z = \lambda + i\mu$, $\lambda, \mu \in \mathbb{R}$

A symmetric, $(A-z = (A-\lambda) - i\mu$

$$\boxed{\| (A-z) u \|^2} = \langle (A-z) u, (A-z) u \rangle$$

$$= \langle (A-\lambda) u, (A-\lambda) u \rangle$$

$$+ i\mu \langle u, (A-\lambda) u \rangle$$

$$\qquad \shortparallel$$

$$- i\mu \langle (A-\lambda) u, u \rangle + \langle \mu u, \mu u \rangle$$

$$\boxed{= \| (A-\lambda) u \|^2 + \mu^2 \| u \|^2 \geq \mu^2 \| u \|^2 \quad (2)}$$

So if $\operatorname{Im} z > 0$ (or $\operatorname{Im} z < 0$) $\Rightarrow \ker(A-z) = \{0\}$.

If also $\operatorname{Ran}(A-z) = \mathcal{H}$, then $(A-z)^{-1}$ is defined on all $\mathcal{H}$ and by (2), with

$u = (A-z)^{-1}\psi$

$$\|(A-z)^{-1}\psi\| = \|u\| \underset{(2)}{\leq} \frac{1}{|\operatorname{Im} z|}\|(A-z)u\|$$

$$= \frac{1}{|\operatorname{Im} z|}\|\psi\| \quad \forall \psi \in \mathcal{H}$$

$\boxtimes$

<u>Def. 12</u> (Spectrum, resolvent set etc)

Let A be a closed op. on $\mathcal{H}$. The resolvent set

$\rho = \rho(A) = \{z \in \mathbb{C} \mid A - z \text{ is invertible}\}$

i.e., for those $z \in \mathbb{C}$: $\operatorname{Ran}(A-z) = \mathcal{H}$,

$\operatorname{Ker}(A-z) = \{0\}$, and $(A-z)^{-1}$ is a bounded op..

We call $R_z = R(z) = (A-z)^{-1}$ the resolvent of A.

The spectrum of A, $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

<u>Remark</u>: By Lemma 12, if A is self-adj., then

$\rho(A) \supset \mathbb{C} \setminus \mathbb{R}$, so $\sigma(A) \subset \mathbb{R}$!

<u>Ex</u>: • x_j = Mult. by a coordinate, $\sigma(x_j) = \mathbb{R}$

• p_j : momentum, $\sigma(p_j) = \mathbb{R}$

2 Facts about resolvents (at least for self-adj. op. A)

1) $$A \frac{1}{A-z} = \frac{1}{A-z} A \qquad (3)$$

or, more precisely, $\forall \psi \in \mathcal{D}(A)$

$$A \frac{1}{A-z} \psi = \frac{1}{A-z} A \psi \qquad \forall z \in \rho(A)$$

i.e., (3) holds on the domain $\mathcal{D}(A)$

(note that $A \frac{1}{A-z}$ is always defined on all of $\mathcal{H}$, and by the spectral theorem

$$\left\| A \frac{1}{A-z} \right\| \leq \sup_{x \in \mathbb{R}} \left| x \frac{1}{x-z} \right| < \infty$$

2) Resolvents commute, i.e., $\forall z_1, z_2 \in \rho(A)$,

$$\frac{1}{A-z_1} \frac{1}{A-z_2} = \frac{1}{A-z_2} \frac{1}{A-z_1}$$

since

$$\frac{1}{A-z_1} - \frac{1}{A-z_2} = \frac{1}{A-z_1} \left(A - z_2 - (A - z_1) \right) \frac{1}{A-z_2}$$

$$= - \frac{1}{A-z_1} (z_1 - z_2) \frac{1}{A-z_2} = - (z_1 - z_2) \frac{1}{A-z_1} \frac{1}{A-z_2}$$

3) $$\left(\frac{1}{A-z} \right)^* = \frac{1}{A-\bar{z}} \quad \text{if } A^* = A.$$

I.2.5 Dynamics

In this section, we construct the exponential e^{-itA} for a self-adjoint operator A, which allows us to solve the abstract Schrödinger equation

$$\boxed{i\frac{\partial \psi}{\partial t} = A\psi \qquad (4)}$$

where $\psi : t \mapsto \psi(t) \in \mathcal{H}$ is a path in $\mathcal{H}$

A self-adj. and $\psi(0) = \psi|_{t=0} = \psi_0$

is the initial condition.

Thus self-adjointness of A implies the existence of dynamics!

Theorem 13 If A is self-adj., then there is a unique family of operators $U(t) = e^{-itA}$ with the following properties:

for all $s, t \in \mathbb{R}$

a) $i\frac{\partial}{\partial t} U(t) = A\,U(t) = U(t)\,A$

b) $\mathcal{U}(0) = \mathbb{1}$, $\mathcal{U}(t)\psi \to \psi$ for $t \to 0$

c) $\mathcal{U}(t)\,\mathcal{U}(s) = \mathcal{U}(t+s)$

d) $\|\mathcal{U}(t)\psi\| = \|\psi\|$.

Proof: We will define the exponential e^{iA} for an unbounded, self-adj op. A by approximating A by bounded operators and then using the usual power series def. of the exponential for bounded operators.

If B is bounded,
$$e^{B} := \sum_{n=0}^{\infty} \frac{B^n}{n!} \quad , \; B^0 = \mathbb{1}, \; B^{n+1} = B\,B^n$$
which converges absolutely in the operator norm, since
$$\sum_{n=0}^{\infty} \frac{\|B^n\|}{n!} \le \sum_{n=0}^{\infty} \frac{\|B\|^n}{n!} = e^{\|B\|} < \infty .$$

Lemma 14. If A is bounded and symmetric and $\mathcal{U}(t) := e^{-itA}$, then assertions a), b), c), d) in Thm 13 hold.

proof: see homework, it is done similarly to the usual discussion of the exponential!

A small discussion on the side:

Let $x \in \mathbb{R}$, $\lambda > 0$,

$$f_\lambda(x) := \frac{1}{2}\lambda^2\left(\frac{1}{x+i\lambda} + \frac{1}{x-i\lambda}\right)$$

$$= \frac{1}{2}\lambda^2 \frac{2x}{(x+i\lambda)(x-i\lambda)}$$

$$= \frac{\lambda^2 x}{x^2+\lambda^2} \to x \quad \lambda \to \infty$$

and note that for fixed $\lambda > 0$,

$\|f_\lambda\|_\infty = \lambda$, so f_λ is bounded.

and it approximates the identity.

Idea: use f_λ to regularize A, if A is an unbounded self-adj. operator and hope that

$f_\lambda(A) \to A$ in some sense.

So define

$$A_\lambda := \frac{1}{2}\lambda^2\left(\frac{1}{A+i\lambda}+\frac{1}{A-i\lambda}\right), \lambda>0.$$

is symmetric if $A=A^*$ and bounded.

Note also

$$\boxed{A_\lambda = B_\lambda A \qquad \text{on } \mathcal{D}(A) \qquad (4)}$$

where

$$B_\lambda = \frac{1}{2} i\lambda\left((A+i\lambda)^{-1}-(A-i\lambda)^{-1}\right) \qquad (5)$$

and

$$\mathbb{1} - B_\lambda = \frac{1}{2}\left((A+i\lambda)^{-1}+(A-i\lambda)^{-1}\right)A \qquad (6)$$

on $\mathcal{D}(A)$.

Indeed,

$$A_\lambda = \frac{1}{2}\lambda^2 \quad (A+i\lambda)^{-1}\left(A-i\lambda+A+i\lambda\right)(A-i\lambda)^{-1}$$

$$= \lambda^2 (A+i\lambda)^{-1} A (A-i\lambda)^{-1}$$

$$\underset{\text{on } \mathcal{D}(A)}{=} \lambda^2 \underbrace{(A+i\lambda)^{-1}(A-i\lambda)^{-1}} A$$

note $(A+i\lambda)^{-1}-(A-i\lambda)^{-1}$

$$= (A+i\lambda)^{-1}\left(A-i\lambda-(A+i\lambda)\right)(A-i\lambda)^{-1}$$

$$= -2i\lambda\,(A+i\lambda)^{-1}(A-i\lambda)^{-1}$$

so

$$A_\lambda = \frac{i}{2}\lambda\,\left(\left(A+i\lambda\right)^{-1}-\left(A-i\lambda\right)^{-1}\right)A$$

on $\mathcal{D}(A)$., i.e., (4) holds

Also

$$\mathbb{1} - B_\lambda = \mathbb{1} - \frac{i\lambda}{2}\left((A+i\lambda)^{-1}-(A-i\lambda)^{-1}\right)$$

$$= \frac{1}{2}\left(\mathbb{1} - i\lambda\,(A+i\lambda)^{-1}\right) + \frac{1}{2}\left(\mathbb{1} + i\lambda\,(A-i\lambda)^{-1}\right)$$

$$= \frac{1}{2}(A+i\lambda)^{-1}(A+i\lambda-i\lambda) + \frac{1}{2}(A-i\lambda)^{-1}(A-i\lambda+i\lambda)$$

$$= \frac{1}{2}\left((A+i\lambda)^{-1}+(A-i\lambda)^{-1}\right)A.$$

so (6) holds