Lecture 4. Mathematical Physics SS 2013

Note Title 4/18/2013

Recall from last time

<u>Theorem 13</u> If A is self-adj., then there is a unique family of operators $U(t) = e^{-itA}$ with the following properties:

for all $s, t \in \mathbb{R}$

a) $i \frac{\partial}{\partial t} U(t) = A U(t) = U(t) A$

b) $U(0) = \mathbb{1}$, $U(t)\psi \to \psi$ for $t \to 0$

c) $U(t) U(s) = U(t+s)$

d) $\|U(t)\psi\| = \|\psi\|$.

For the proof we noted that if A is bounded & symmetric, then all the above hold:

<u>Lemma 14.</u> If A is bounded and symmetric and $U(t) := e^{-itA}$, then assertions a), b), c), d), in Thm 13 hold.

One simply <u>defines</u> $U(t) = \sum_{n=0}^{\infty} \frac{(-itA)^n}{n!}$

which is a norm convergent absolutely summable series and check that this does the job!

Continuing the proof of Thm 13.

Last time we introduced

$$A_\lambda := \frac{1}{2}\lambda^2\left(\frac{1}{A+i\lambda}+\frac{1}{A-i\lambda}\right), \quad \lambda>0.$$

is symmetric if $A=A^*$ and bounded.

Note also

$$\boxed{A_\lambda = B_\lambda A \quad \text{on } \mathcal{D}(A) \qquad (4)}$$

where

$$B_\lambda = \frac{1}{2} i\lambda\left((A+i\lambda)^{-1}-(A-i\lambda)^{-1}\right) \qquad (5)$$

and

$$\mathbb{1} - B_\lambda = \frac{1}{2}\left((A+i\lambda)^{-1}+(A-i\lambda)^{-1}\right)A \qquad (6)$$

on $\mathcal{D}(A)$.

and checked that (4), (5), and (6) hold.

Lemma 15 $\forall \lambda > 0$

$\|B_\lambda\| \le 1$. $\forall \phi \in \mathcal{H}$ $\|(\mathbb{1} - B_\lambda)\phi\| \to 0$ $\lambda \to \infty$

and for $\psi \in \mathcal{D}(A)$ $\lim_{\lambda \to \infty} A_\lambda \psi = A\psi$.

proof:

$$\|B_\lambda\| \le \frac{\lambda}{2}\left(\|(A+i\lambda)^{-1}\| + \|(A-i\lambda)^{-1}\|\right) \le \frac{\lambda}{\lambda} = 1.$$

Let $\psi \in \mathcal{D}(A)$ then

$$(\mathbb{1} - B_\lambda)\psi = \frac{1}{2}\left((A+i\lambda)^{-1} + (A-i\lambda)^{-1}\right)A\psi$$

so

$$\|(\mathbb{1} - B_\lambda)\psi\| \le \frac{1}{\lambda}\|A\psi\| \to 0 \quad \lambda \to \infty$$

Thus

$$\lim_{\lambda \to \infty} (\mathbb{1} - B_\lambda)\psi = 0 \qquad (7)$$

for a dense set of ψ (including $\mathcal{D}(A)$).

Also $\|\mathbb{1} - B_\lambda\| \le \|\mathbb{1}\| + \|B_\lambda\| \le 2$ all λ.

So let $\varepsilon > 0$ and $\phi \in \mathcal{H}$ and choose $\psi \in \mathcal{D}(A)$

s.t. $\|\phi - \psi\| \le \frac{\varepsilon}{2}$. Then

$$(\mathbb{1} - B_\lambda)\phi = \underbrace{(\mathbb{1} - B_\lambda)\psi}_{\to 0 \text{ as } \lambda \to \infty} + \underbrace{(\mathbb{1} - B_\lambda)(\phi - \psi)}_{\|\,\cdot\,\| \le 2\|\phi - \psi\| \le \varepsilon}$$

$$\Rightarrow \limsup_{\lambda \to \infty} \|(\mathbb{1} - B_\lambda)\phi\| \le \varepsilon \qquad \forall \varepsilon > 0$$

$$\Rightarrow \lim_{\lambda \to \infty} (\mathbb{1} - B_\lambda)\phi = 0.$$

For the last:

We have $B_\lambda \psi \to \psi \quad \lambda \to \infty \quad \forall \psi \in \mathcal{H}$

So if $\psi \in \mathcal{D}(A)$

$$A_\lambda \psi = B_\lambda A \psi \to A\psi \quad \text{as } \lambda \to \infty$$

$\boxed{}$

Continuing with the proof of Thm 13:

A_λ is bdd & symmetric.

$\rightsquigarrow$ define $\displaystyle e^{-itA_\lambda} = \sum_{n=0}^{\infty} \frac{(-itA_\lambda)^n}{n!}$

Claim: $\{e^{-itA_\lambda}\}_{\lambda > 0}$ is strongly a Cauchy family

i.e., $\|(e^{-itA_{\lambda'}} - e^{-itA_\lambda})\psi\| \to 0$ as $\lambda, \lambda' \to \infty$

for each $\psi \in \mathcal{H}$

Write A_λ for $-tA_\lambda$ in the following:

Note

$$e^{iA_\lambda} - e^{iA_\lambda} = \int_0^1 \frac{\partial}{\partial s} e^{isA_\lambda} e^{i(1-s)A_\lambda} ds$$

$$= i \int_0^1 e^{isA_{\lambda'}} (A_{\lambda'} - A_\lambda) e^{i(1-s)A_\lambda} ds$$

$$= i \int_0^1 e^{isA_{\lambda'}} e^{i(1-s)A_\lambda} (A_{\lambda'} - A_\lambda) ds$$

$A_{\lambda'}$ and A_λ commute !

So for $\psi \in \mathcal{D}(A)$ we have

$$\| (e^{iA_{\lambda'}} - e^{iA_\lambda}) \psi \| \leq \| i \int_0^1 e^{isA_{\lambda'}} e^{i(1-s)A_\lambda} (A_{\lambda'} - A_\lambda) \psi \, ds \|$$

$$\leq \int_0^1 \underbrace{\| e^{isA_{\lambda'}} e^{i(1-s)A_\lambda} (A_{\lambda'} - A_\lambda) \psi \|}_{= \| (A_{\lambda'} - A_\lambda) \psi \|} dt$$

$$= \| (A_{\lambda'} - A_\lambda) \psi \| \to 0 \quad \lambda', \lambda \to \infty \qquad (8)$$

So for $\psi \in \mathcal{D}(A)$

$$\boxed{e^{iA} \psi := \lim_{\lambda \to \infty} e^{iA_\lambda} \psi \quad \text{exists}} \qquad (9)$$

Also $\| e^{iA} \psi \| = \lim_{\lambda \to \infty} \underbrace{\| e^{iA_\lambda} \psi \|}_{= \|\psi\|} = \|\psi\|$

holds, so can extend

e^{iA} to all of $\mathcal{H}$ since $\mathcal{D}(A)$ is dense in $\mathcal{H}$.

and e^{iA} and e^{iA_λ} are unif. bdd.

Now replace A by $-tA$, so we have defined

$$U(t) = e^{-itA} \quad \text{on all of } \mathcal{H}$$

and $\| U(t) \| = 1$.

We also have

$$U(t)\, U(s) \phi = \lim_{\lambda \to \infty} e^{-itA_\lambda} e^{-isA_\lambda} \phi$$

$$= \lim_{\lambda \to \infty} e^{-i(t+s)A_\lambda} \phi$$

$$= U(t+s)\phi$$

hence $U(t)\, U(s) = U(t+s)$ and c) holds.

Trivially $U(0) = \mathbb{1}$.

Note also that $t \mapsto e^{-itA_\lambda} \phi$ is cont. in t

$\forall \phi \in \mathcal{H}$ (it is even cont. in the operator norm)

Now an argument identical to (8) yields

$$\| (e^{-itA_{\lambda'}} - e^{-itA_\lambda}) \phi \|$$

$$\leq |t| \, \| (A_{\lambda'} - A_\lambda) \phi \|$$

So $e^{-iA_\lambda} \phi$ converges locally uniformly to $\mathcal{U}(t)\phi$

for all $\phi \in \mathcal{D}(A)$

$\Rightarrow$ $t \mapsto \mathcal{U}(t)\phi$ is contin. for all $\phi \in \mathcal{D}(A)$.

Now let $\psi \in \mathcal{H}$. and note

$$\| (\mathcal{U}(t) - \mathcal{U}(t_0)) \psi \| \leq \| (\mathcal{U}(t) - \mathcal{U}(t_0))(\psi - \phi) \|$$

$$+ \| (\mathcal{U}(t) - \mathcal{U}(t_0)) \phi \|$$

$$\leq 2 \| \psi - \phi \| + \| (\mathcal{U}(t) - \mathcal{U}(t_0)) \phi \|$$

So if $\varepsilon > 0$, $\phi \in \mathcal{D}(A)$ with $\| \psi - \phi \| < \frac{\varepsilon}{4}$

and $\delta > 0$ s.t. $\| (\mathcal{U}(t) - \mathcal{U}(t_0)) \phi \| < \frac{\varepsilon}{2}$

Then

$$\| (\mathcal{U}(t) - \mathcal{U}(t_0)) \psi \| \leq 2 \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $\mathcal{U}(t)\psi$ is cont. at t_0. (for any $t_0 \in \mathbb{R}$!)

i.e., $t \mapsto U(t)\psi$ is cont. $\forall \psi \in \mathcal{H}$!

Note also that

$$i \frac{\partial}{\partial t} e^{-itA_\lambda} = A_\lambda e^{-itA_\lambda} = e^{-itA_\lambda} A_\lambda$$

So $$e^{-itA_\lambda} - e^{-it_0 A_\lambda} = \int_{t_0}^{t} e^{-isA_\lambda} A_\lambda \, ds$$

and for $\phi \in \mathcal{H}$

$$i\left(e^{-itA_\lambda} - e^{-it_0A_\lambda}\right)\phi = \int_{t_0}^{t} e^{-isA_\lambda} A_\lambda \phi \, ds$$

also note

$$\int_{t_0}^{t} e^{-isA_\lambda} A_\lambda \phi \, ds \to \int_{t_0}^{t} U(s) A\phi \, ds$$

$\forall \phi \in \mathcal{D}(A)$

(see homework), so

$$\boxed{i(U(t) - U(t_0))\phi = \int_{t_0}^{t} U(s) A \phi \, ds \qquad (10)} \quad \forall \phi \in \mathcal{D}(A)$$

First for all t, t_0, but then also for all $t \in \mathbb{R}$

Since $s \mapsto U(s) A\phi$ is cont., the Fund. thm.

of Analysis 1 shows

$$\lim_{t \to t_0} \frac{i\,(\mathcal{U}(t)\phi - \mathcal{U}(t_0)\phi)}{t - t_0} = \mathcal{U}(t_0) A \phi$$

i.e., $t \mapsto \mathcal{U}(t)\phi$ is diff. whenever $\phi \in \mathcal{D}(A)$

and a) holds

$\boxed{\text{}}$

<u>Remark</u>. We did not show $i\partial_t \mathcal{U}(t) = A\, \mathcal{U}(t)$

i.e., $\mathcal{U}(t)\, \mathcal{D}(A) \subset \mathcal{D}(A)$ and