Lecture 6. Mathematical Physics SS 2013

Note Title 4/18/2013

Def 20: • V, H_0 as in Thm 19. V is infinitesimally H_0 small if the relative bound of V w.r.t. H_0 is 0.

• We say that a potential $V : \mathbb{R}^d \to \mathbb{R}$ is a Kato potential if V is infinitesimally $-\Delta$-bdd.

Remarks • The set of all Kato potentials is a vector space.

• If $P : \mathbb{R}^d \to \mathbb{R}^d$ is a projection onto a n-dim. subspace of $\mathbb{R}^d$ and if $V : \mathbb{R}^n \to \mathbb{R}$ is a Kato pot., then also $\tilde{V} : \mathbb{R}^d \to \mathbb{R}$, $\hat{V}(x) := V(Px)$ $x \in \mathbb{R}^d$, is a Kato potential.

Ex: $\mathbb{R}^d \ni x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^{d-n}$

$Px = x_1$. $\hat{V}(x) = V(x_1) = V(Px)$.

Have: $\forall \alpha > 0 \ \exists b < \infty$ s.t.

$$\|V\varphi\|^2_{L^2(\mathbb{R}^n)} \le \alpha^2 \|-\Delta_1 \varphi\|^2_{L^2(\mathbb{R}^n)} + b^2 \|\varphi\|^2_{L^2(\mathbb{R}^n)}$$

Take $\tilde{\varphi} \in D(\Delta) \subset L^2(\mathbb{R}^d)$, note that the Laplacian splits $\Delta = \Delta_1 + \Delta_2$

or better $\Delta = \Delta_1 \otimes \mathbb{1}_{d-n} + \mathbb{1}_n \otimes \Delta_2$

Laplace on $\mathbb{R}^n$ Laplace on $\mathbb{R}^{d-n}$

Identities on $\mathbb{R}^{d-n}$, resp. $\mathbb{R}^n$.

$$\|\tilde{V}\tilde{\varphi}\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{d-n}} \underbrace{\left(\int_{\mathbb{R}^n} V(x_1)^2 |\tilde{\varphi}(x_1,x_2)|^2 dx_1\right)}_{\substack{\leq \alpha^2 \int |-\Delta_1 \varphi(x_1,x_2)|^2 dx_1 \\ + b^2 \int |\varphi(x_1,x_2)|^2 dx_1}} dx_2$$

$$\leq \alpha^2 \|-\Delta_1 \tilde{\varphi}\|^2_{L^2(\mathbb{R}^d)} + b^2 \|\varphi\|_{L^2(\mathbb{R}^d)}$$

$$\leq \alpha^2 \|-\Delta \tilde{\varphi}\|^2_{L^2(\mathbb{R}^d)} + b^2 \|\varphi\|_{L^2(\mathbb{R}^d)}$$

$$-\Delta_1 = \sum_{j=1}^{n} P_j^2 \leq \sum_{j=1}^{d} P_j^2 \quad \text{since } P_j^2 \geq 0,$$

Altogether: If $P_e : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ and $P_{j,l} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ are projections onto 3-dimensional subspaces of $\mathbb{R}^{3N}$ and $V_e : \mathbb{R}^3 \to \mathbb{R}$, $V_{j,l} : \mathbb{R}^3 \to \mathbb{R}$ are Kato-potentials, then

$$V(x) := \sum_{\ell} V_\ell (P_\ell x) + \sum_{j,s} V_{js} (P_{js} x)$$

is a Kato potential on $L^2(\mathbb{R}^{3N})$.

We want to take $V_\ell = \frac{1}{|x|}$ and $P_\ell : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$

$x \mapsto x_\ell$ and $P_{js}(x) = x_j - x_s$

$x = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$.

<u>Thm 21</u> Any $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$

is a Kato potential on $L^2(\mathbb{R}^3)$.

<u>Remark:</u> • $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$

$\iff \exists\, V_1 \in L^2(\mathbb{R}^3),\ V_2 \in L^\infty(\mathbb{R}^3)$ s.t.

$$V = V_1 + V_2$$

• $V(x) = \frac{1}{|x|} = V_1(x) + V_2(x)$ with

$$x \mapsto V_1(x) := \begin{cases} \frac{1}{|x|} & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \in L^2(\mathbb{R}^3)$$

$$x \mapsto V_2(x) := \begin{cases} 0 & |x| \le 1 \\ \frac{1}{|x|} & |x| > 1 \end{cases} \in L^\infty(\mathbb{R}^3).$$

So all this together shows that the Hamiltonian with Coulomb potentials is s.a. on $H^2(\mathbb{R}^{3N})$! $\square$

<u>Proof of thm:</u> Since any pot. $V_2 \in L^\infty(\mathbb{R}^d)$ is a bdd. multiplication operator, we only have to worry about $V \in L^2(\mathbb{R}^3)$!

Take $\varphi \in \mathcal{D}(-\Delta) = H^2(\mathbb{R}^3)$, $M > 0$.

Set $A_M := \{x \in \mathbb{R}^3 \mid |V(x)| \geq M\}$.

Then

$$\|V\varphi\|_2^2 = \int_{\mathbb{R}^3} V(x)^2 |\varphi(x)|^2 dx$$

$$= \int_{A_M} V(x)^2 |\varphi(x)|^2 dx + \underbrace{\int_{A_M^c} \underbrace{V(x)^2}_{\leq M \text{ on } A_M^c} |\varphi(x)|^2 dx}_{\leq M^2 \|\varphi\|_2^2}.$$

$$\leq \underbrace{\left(\int_{A_M} |V(x)|^2 dx\right)}_{\to 0 \text{ as } M \to \infty \text{ since } V \in L^2.} \|\varphi\|_\infty^2 + M^2 \|\varphi\|_2^2$$

So need a bound on $\|\varphi\|_\infty$!

Let $\varphi \in \mathcal{S}$ (Schwartz class), then

$$|\varphi(x)|^2 = \left|\int e^{2\pi i x \xi} \hat{\varphi}(\xi) d\xi\right|^2 \leq \left(\int |\hat{\varphi}(\xi)| d\xi\right)^2$$

$$= \left(\int (1+|2\pi\xi|^2)^{-1} (1+|2\pi\xi|^2) \, |\hat{\varphi}(\xi)| \, d\xi \right)^2$$

$$\underset{C.S.I}{\leq} \underbrace{\int (1+|2\pi\xi|^2)^{-2} d\xi}_{<\infty} \; \underbrace{\int (1+|2\pi\xi|^2)^2 |\hat{\varphi}(\xi)|^2 d\xi}_{= \|(I-\Delta)\varphi\|_2^2}$$

$$\Rightarrow \|\varphi\|_\infty^2 \leq \underbrace{\int (1+|2\pi\xi|^2)^{-2} d\xi} \; \|(I-\Delta)\varphi\|_2^2$$

$$\Rightarrow \|V\varphi\|_2^2 \leq \int_{|V|>M} |V(x)|^2 dx \int (1+(2\pi\xi)^2)^{-2} d\xi \cdot$$

$$\cdot \|(I-\Delta)\varphi\|_2^2 + M^2 \|\varphi\|_2^2$$

$\boxed{\text{QED}}$

<u>Remarks</u> $P = -i\nabla$, $P^2 = -\Delta$. Let V be a Kato pot., then the kinetic energy bounds the total energy and vice versa.

If $H = P^2 + V$, then $\exists$ $c, d, \tilde{c}, \tilde{d}$ s.t.

$$\|H\varphi\| \leq c\|P^2\varphi\| + d\|\varphi\| \quad \forall \varphi \in H^2$$

$$\|P^2\varphi\| \leq \tilde{c}\|H\varphi\| + \tilde{d}\|\varphi\| \quad \forall \varphi \in H^2.$$

See H.W.

I.2.7. The harmonic oscillator

We will concentrate on $d=1$.

$$H_{osc} = \frac{P^2}{2m} + \frac{m\omega^2}{2} x^2 \qquad P = -i\hbar\partial_x$$

$$= \frac{-\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2$$

Rem: A scaling argument: Set

$$U_\lambda \psi(x) = \lambda^{-\frac{1}{2}}\psi\left(\frac{x}{\lambda}\right) = \psi_\lambda(x)$$

and note that this is unitary on $L^2(\mathbb{R})$.

Also note

$$(\partial_x U_\lambda\psi)(x) = \partial_x \lambda^{-1/2}\psi\left(\frac{x}{\lambda}\right) = \lambda^{-1}\lambda^{-1/2}(\psi')\left(\frac{x}{\lambda}\right)$$

$$= \lambda^{-1} U_\lambda(\partial_x\psi)(x)$$

So with $P = -i\hbar\partial_x$, we have

$$P U_\lambda \psi = \lambda^{-1} U_\lambda P \quad \text{or}$$

$$\boxed{U_\lambda^{-1} P U_\lambda = \lambda^{-1} P}$$

and if V = multiplication by $V(x)$, then

$$(V U_\lambda\psi)(x) = V(x)(U_\lambda\psi)(x) = V(x)\lambda^{-1/2}\psi\left(\frac{x}{\lambda}\right)$$

$$= \lambda^{-1/2} \left(V(\lambda\cdot)\psi\right)\left(\frac{x}{\lambda}\right)$$

$$= \left(\mathcal{U}_\lambda \left(V(\lambda\cdot)\psi\right)\right)(x)$$

So $\mathcal{U}_\lambda^{-1} V \mathcal{U}_\lambda = V(\lambda\cdot)$. In particular, if $V(x) = \frac{m\omega^2}{2} x^2$, then

$$\mathcal{U}_\lambda^{-1} V \mathcal{U}_\lambda = \lambda^2 \frac{m\omega^2}{2} x^2.$$

$$\Rightarrow \mathcal{U}_\lambda^{-1} H_{osc} \mathcal{U}_\lambda = \frac{-\hbar^2}{2m\lambda^2} \partial_x^2 + \lambda^2 \frac{m\omega^2}{2} x^2$$

choose $\lambda > 0$ s.t. $\frac{\hbar^2}{2m\lambda^2} = \lambda^2 \frac{m\omega^2}{2}$

or $\lambda^4 = \frac{\hbar^2}{m^2\omega^2} \Leftrightarrow \lambda^2 = \frac{\hbar}{m\omega}$

For this value of λ, one gets

$$\lambda^2 m\omega^2 = \frac{\hbar}{m\omega^2} m\omega^2 = \hbar\omega$$

$$\frac{\hbar^2}{m\lambda^2} = \frac{\hbar^2}{m\frac{\hbar}{m\omega}} = \hbar\omega$$

$$\Rightarrow \mathcal{U}_\lambda^{-1} H_{osc} \mathcal{U}_\lambda = \hbar\omega \underbrace{\left(-\frac{1}{2}\partial_x^2 + \frac{x^2}{2}\right)}_{= H_{osc} \text{ with } \hbar = m = \omega = 1 \,!}$$

So H_{osc} is unitary equivalent to

$\hbar\omega$ times an harmonic oscillator with

$$\boxed{\hbar = m = \omega = 1.}$$

$\Rightarrow$ Focus on studying $H_{osc} = -\frac{1}{2}\partial_x^2 + x^2$

$$= \frac{P^2}{2} + \frac{x^2}{2}$$

with $P = -i\partial_x$ $(\hbar = 1)$.

Domain of $H_{osc} = \mathcal{D}(P^2) \cap \mathcal{D}(x^2)$

$$= H^2(\mathbb{R}) \cap \mathcal{D}(x^2)$$

Space of Schwartz class functions

$$\mathcal{S} := \{\varphi \in C^\infty(\mathbb{R}) \mid \|\langle x\rangle^\alpha P^\beta \varphi\|_2 < \infty \ \forall \alpha, \beta \in \mathbb{N}_0\}$$

$$= \{\varphi \in C^\infty(\mathbb{R}) \mid \|\langle x\rangle^\alpha P^\beta \varphi\|_\infty < \infty, \forall \alpha, \beta \in \mathbb{N}_0\}$$

where $\langle x\rangle = (1+x^2)^{1/2}$.

Easy: H_{osc} on $\mathcal{S}$ (and on $H^2 \cap \mathcal{D}(x^2)$) is symmetric (int. by parts!)

<u>Fact:</u> H_{osc} on $\mathcal{S}$ has a unique self-adj. extension, i.e., H_{osc} on $\mathcal{S}$ is essentially s.a..