Lecture 7. Mathematical Physics SS 2013

Note Title 4/18/2013

Consider the harmonic oscillator

$$H_{osc} = \frac{1}{2}(P^2 + x^2)$$

on $\mathcal{D}(P^2) \cap \mathcal{D}(x^2) = H^2(\mathbb{R}) \cap \mathcal{D}(x^2) \supset \mathcal{S}$

Schwartz class.

Claim: H_{osc} is essentially self-adjoint on $\mathcal{S}$

(i.e., it has only one self-adjoint extension).

We will use the

Fundamental criterion for essential self-adjointness

Let $H : \mathcal{D} \to \mathcal{H}$.

a) $H : \mathcal{D} \to \mathcal{H}$ is esst. s.a.

b) $\ker(H^* \mp i) = \{0\}$

c) $\mathrm{Ran}(H \mp i)$ is dense in $\mathcal{H}$

Then a) $\Leftrightarrow$ b) $\Leftrightarrow$ c).

Corollary 22: Assume $H : \mathcal{D} \to \mathcal{H}$

and for $n \in \mathbb{N}$, $\exists\, \varphi_n \in \mathcal{D}$, $E_n \in \mathbb{R}$ with

$$H \varphi_n = E_n \varphi_n$$

and $\mathcal{D}_1 = \text{span}\{\varphi_n \mid n \in \mathbb{N}\}$ is dense in $\mathcal{H}$.

Then H is <u>essentially</u> s.a. on $\mathcal{D}_1$. and
$\sigma(H) = \bigcup_{n \in \mathbb{N}} \{E_n\}$.

<u>Remark</u>: If H is symmetric on $\mathcal{D} \supset \mathcal{D}_1$, then H is also essent. s.a. on $\mathcal{D}$.

<u>proof:</u> We will check part b) of the fund. criterion for essential s.a.

Assume $f \in \mathcal{D}(H^*)$ with $H^* f = \pm i f$.

Then

$$\mp i \langle f, \varphi_n \rangle = \langle \pm i f, \varphi_n \rangle = \langle H^* f, \varphi_n \rangle$$

$$= \langle f, H \varphi_n \rangle = E_n \langle f, \varphi_n \rangle$$

$$\Rightarrow \underbrace{(E_n \mp i)}_{\neq 0 \text{ since } E_n \in \mathbb{R}} \langle f, \varphi_n \rangle = 0$$

so $\langle f, \varphi_n \rangle = 0 \quad \forall\, n \in \mathbb{N}$

$\Rightarrow \langle f, v \rangle = 0 \quad \forall\, v \in \text{span}(\varphi_n \mid n \in \mathbb{N}) = \mathcal{D}_1$

since $\mathcal{D}_1$ is dense in $\mathcal{H} \Rightarrow f = 0$ so b) holds.

Let $\bar{H}$ = s.a. ext. of H = closure of H. also has φ_n as eigenvectors and E_n as eigenvalues, so

$$\underbrace{\bigcup_{n\in\mathbb{N}} \{E_n\}}_{} \subset \underbrace{\sigma(\bar{H})}_{\text{is closed}}$$

$$\Rightarrow \quad \overline{\bigcup_{n\in\mathbb{N}} \{E_n\}} \subset \overline{\sigma(\bar{H})} = \sigma(\bar{H}).$$

On the other hand, if $z \in \overline{\bigcup_{n\in\mathbb{N}} \{E_n\}}$, then

$$R_z := \sum_{n\in\mathbb{N}} \frac{1}{E_n - z} |\varphi_n\rangle\langle\varphi_n| \qquad \text{(for physicists)}$$

$$= \sum_{n\in\mathbb{N}} \frac{1}{E_n - z} \langle \varphi_n, \cdot \rangle \varphi_n$$

is defined and one checks that R_z is the resolvent of $\bar{H}$.

Thus $\rho(\bar{H}) \supset \overline{\bigcup_{n\in\mathbb{N}} \{E_n\}}^c$, so

$$\sigma(\bar{H}) = \rho(\bar{H})^c \subset \overline{\bigcup_{n\in\mathbb{N}} \{E_n\}}.$$

$\boxtimes$

<u>Def 23.</u> (creation and annihilation operators)

Let $p = -i\partial_x$. Set

$$a := \frac{1}{\sqrt{2}}(x + ip)$$

$$a^* = \frac{1}{\sqrt{2}}(x - ip) \quad \text{the adjoint of } a.$$

(at least defined on $\mathcal{S}$).

• Commutator $[A, B] := AB - BA$

(one should be careful here with the domains ....!)

Lemma 24 a) $[a, a^*] = 1$.

b) Let $H_{osc} = \frac{1}{2}(p^2 + x^2)$, then

$$H_{osc} = a^* a + \frac{1}{2} \quad \text{(at least on } \mathcal{S}\text{)}.$$

c) $$\sigma(H_{osc}) = \bigcup_{n \in \mathbb{N}_0} \{ n + \tfrac{1}{2} \}$$

with eigenfunctions

$$\psi_0 := (2\pi)^{-1/4} e^{-\frac{x^2}{2}}$$

$$\psi_n := \frac{1}{\sqrt{n!}} (a^*)^n \psi_0 \qquad n \in \mathbb{N}.$$

Pf: a) + b) see HW.

c): Define the particle number operator

$$\boxed{N := a^* a}$$

So $H_{osc} = N + \frac{1}{2}$

First we find the ground state and the ground state energy of H_{osc}.

Note $\langle \varphi, N\varphi \rangle = \langle \varphi, a^* a \varphi \rangle$

$$= \langle a\varphi, a\varphi \rangle = \| a\varphi \|^2 \geq 0.$$

$$\Rightarrow \quad \langle \varphi, H_{os} \varphi \rangle = \| a\varphi \|^2 + \frac{1}{2} \| \varphi \|^2$$

$$\Rightarrow \quad E_0 = \inf \sigma(H_{osc}) = \frac{1}{2}\,.\,!$$

if there exists $\psi_0 \neq 0$ s.t.

$$\boxed{a\,\psi_0 = 0.}$$

$$\Leftrightarrow \qquad (x + ip)\,\psi_0 = 0$$

$$\Leftrightarrow \quad x\,\psi_0 + \psi_0' = 0$$

$$\Rightarrow \quad \psi_0(x) = C_0\, e^{-\frac{x^2}{2}}.$$

To ensure normalization, we set

$$C_0 := (2\pi)^{-1/4}.$$

$$\psi_0(x) = (2\pi)^{-1/4}\, e^{-\frac{x^2}{2}}.$$

<u>Excited states</u>: Recall $[a, a^*] = a a^* - a^* a = 1$.

Note $\boxed{N a} = a^* a a = (a a^* - [a, a^*])\, a$

$$= a a^* a - a = \boxed{a(N-1)} \qquad (1)$$

and

$$\boxed{N a^*} = a^* a a^* = a^*(a^* a + [a, a^*])$$
$$= \boxed{a^*(N+1).} \qquad (2)$$

So by (2): $N a^* \psi_0 = a^*(N+1)\psi_0$
$$= a^* \psi_0$$

and, since by (2)
$$N(a^*)^{n+1} = N a^* (a^*)^n$$
$$= a^*(N+1)(a^*)^n$$
$$= (a^*)^{n+1} + a^* N (a^*)^n$$
$$= (a^*)^{n+1} + a^*\left((a^*)^n + a^* N (a^*)^{n-1}\right)$$
$$= \ldots = (n+1)(a^*)^{n+1} + (a^*)^{n+1} N. \qquad (3)$$

we also have
$$N(a^*)^n \psi_0 = \left(n (a^*)^n + (a^*)^{n+1} N\right)\psi_0$$
$$= n (a^*)^n \psi_0 + 0. \qquad (4)$$

So $\phi_n := (a^*)^n \psi_0$ is an eigenfct. of N

with eigenvalue $n \in \mathbb{N}$, $N\phi_n = n\phi_n$

$\Rightarrow$ ϕ_n, $n \in \mathbb{N}$ is an eigenfct. of H_{osc}.

$$\boxed{H_{osc}\,\phi_n = (n+\tfrac{1}{2})\,\phi_n} \qquad n \in \mathbb{N}_0. \qquad (5)$$

Claim:

$$\|\phi_n\|^2 = \|a^{*n}\psi_0\|^2 = n!.$$

Indeed, note

$$\|\phi_n\|^2 = \langle a^{*n}\psi_0, a^{*n}\psi_0\rangle = \langle a^*\phi_{n-1}, a^*\phi_{n-1}\rangle$$

$$= \langle \phi_{n-1}, a\,a^*\phi_{n-1}\rangle$$

$$= \langle \phi_{n-1}, (a^*a+1)\phi_{n-1}\rangle$$

$aa^* = a^*a+1$

$$= \langle \phi_{n-1}, \underbrace{(N+1)\,\phi_{n-1}}_{=\,n\,\phi_{n-1}}\rangle$$

$$= n\,\langle \phi_{n-1}, \phi_{n-1}\rangle$$

$$= n\cdot(n-1)\langle \phi_{n-2}, \phi_{n-2}\rangle$$

same arg. $= \ldots = n!\,\langle \psi_0, \psi_0\rangle = n!$ ✓

So put $\psi_n := \dfrac{1}{\sqrt{n!}}\,\phi_n = \dfrac{a^{*n}}{\sqrt{n!}}\,\psi_0$.

which is normalized.

Claim: Up to normalization, the ψ_n are the only eigenfunctions of N (and of $H_{osc} = N + \frac{1}{2}$).

pf: Assume ψ is an eigenfct. of N with eigenvalue $\lambda > 0$. Then using (1) inductively (similar to what we did with (2)) one sees

$$N a^m \psi = (\lambda - m) a^m \psi .$$

If we choose m s.t. $\lambda - m < 0$, then

$$\underbrace{\langle a^m \psi, N a^m \psi \rangle}_{\shortparallel} = (\lambda - m) \langle a^m \psi, a^m \psi \rangle$$

$$= \underbrace{(\lambda - m)}_{< 0} \| a^m \psi \|^2 \qquad (6)$$

$$\langle a^m \psi, a^* a \, a^m \psi \rangle$$

$$\shortparallel$$

$$\langle a^{m+1} \psi, a^{m+1} \psi \rangle = \| a^{m+1} \psi \|^2 \geq 0 .$$

So must have $a^m \psi = 0$.

But this implies $a^j \psi = c \psi_0$

where $j \leq m$ is the largest integer s.t. $a^j \psi \neq 0$

and $c \neq 0$ is a constant

But then $\underset{\substack{\shortparallel \\ (\lambda - j)\psi \text{ by } (6)}}{N a^j \psi} = c N \psi_0 = 0$

so must have $\lambda = j$., i.e., ψ corresponds to the eigenvalue j.

If ψ is not proportional to ψ_j, then we can redefine it to satisfy $\langle \psi, \psi_j \rangle = 0$.

But since $\psi_j = \frac{a^{*j}}{\sqrt{n!}} \psi_0$, we get the contradiction

$$0 = \langle \psi, \psi_j \rangle = \frac{1}{\sqrt{n!}} \langle \psi, a^{*j} \psi_0 \rangle$$

$$= \frac{1}{\sqrt{n!}} \langle \underbrace{a^j \psi}_{\substack{\shortparallel \\ c\psi_0}}, \psi_0 \rangle = \frac{c}{\sqrt{n!}} \langle \psi_0, \psi_0 \rangle$$

$$= \frac{c}{\sqrt{n!}} \neq 0.$$

So ψ is a constant times ψ_j. $\boxed{}$