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A Crash Course towards the Spectral Theorem

1st Problem Class on Mathematical Phyiscs · Summer Term 2013

Dipl.-Math. Hans-Jürgen Freisinger

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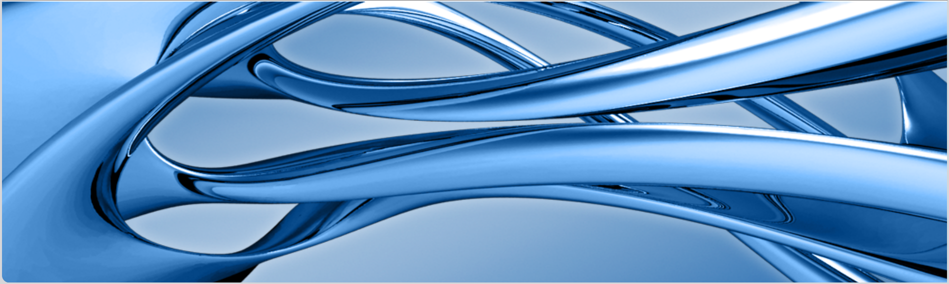
Fakultät für Mathematik, Institut für Analysis



Outline

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Acknowledgement

This talk is based on slides provided by my colleague

Dipl.-Math. Dominik Müller

at GRK 1294 basing on an earlier talk given by him at the University of Ulm.

I would like to thank him for his support.

Literature

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Besides Dominik's talk, the following literature was also helpful:

- Peer Christian Kunstmann, *Spectral Theory*, Notes of a lecture course given in summer 2011.

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- Dirk Werner, *Funktionalanalysis*, Springer, Berlin.

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$$A = \sum_{n=1}^{d} \lambda_n E_n$$

where $E_n x = \langle x, e_n \rangle e_n$.

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$E_n \in \mathscr{L}(\mathbb{C}^d)$ is the orthogonal projection of $\mathbb{C}^d$ onto the eigenspace corresponding to λ_n.

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where $E(\Omega) := \sum_{n=1}^{d} \chi_{\{\lambda_n \in \Omega\}} E_n$ for any $\Omega \in \mathfrak{B}(\mathbb{R})$.

Spectral resolution

$$E(\Omega) := \sum_{n=1}^{d} \chi_{\{\lambda_n \in \Omega\}} E_n, \qquad \Omega \in \mathfrak{B}(\mathbb{R}).$$

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E is called the **spectral resolution** of A.

Spectral measure

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If E is the spectral resolution of A and $\varphi \in \mathbb{C}^d$, then

$$\mu_\varphi(\Omega) := \langle \varphi, E(\Omega)\varphi \rangle$$

defines a complex measure $\mu_\varphi : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathbb{C}$, the s.c. **spectral measure.**

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The spectral theorem for unbounded, selfadjoint operators

If $A : D(A) \subset \mathscr{H} \longrightarrow \mathscr{H}$ is a selfadjont linear operator, there exists a unique spectral resolution $E : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{L}(\mathscr{H})$ such that

$$A = \int_{\sigma(A)} \lambda E(d\lambda).$$

For any $\varphi \in \mathscr{H}$,

$$\mu_\varphi(\Omega) := \langle \varphi, E(\Omega)\varphi \rangle, \qquad \Omega \in \mathfrak{B}(\mathbb{R})$$

defines a complex measure on $\mathfrak{B}(\mathbb{R})$, the s.c. spectral measure.

Unbounded operators

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1. An **operator** A is a linear map $A : D(A) \subset \mathscr{H} \longrightarrow \mathscr{H}$ s. t. the **domain** $D(A)$ is a subspace of $\mathscr{H}$. A is called **densly defined** if $\overline{D(A)} = \mathscr{H}$.

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2. An operator $B : D(B) \longrightarrow \mathscr{H}$ is called an **extension** of A if $D(A) \subset D(B)$ and $A\varphi = B\varphi$ for $\varphi \in D(A)$. We write $A \subset B$.

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3. $A = B \;:\Longleftrightarrow\; A \subset B$ and $B \subset A$.

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- A is not continuous, i.e., unbounded.
- $B := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$ defined on $D(B) = C^1[0,1]$ is an extension of A.

Example: $A := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}, \quad D(A) = C_0^1[0,1], \quad \mathscr{H} = L^2[0,1]$

Moreover for all $\varphi, \psi \in D(A)$,

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Any operator with this property is called **symmetric**.

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Let the operator $A : D(A) \longrightarrow \mathscr{H}$ be densly defined and

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Selfadjointness of an unbounded operator A depends crucially on its domain $D(A)$!!!

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- The **spectrum** of A is then given by $\sigma(A) := \mathbb{C} \setminus \varrho(A)$.

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   - E. g., consider

$$A : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad \varphi \mapsto \varphi''$$

Then $\sigma(A) = [0, \infty)$.

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A roadmap to the spectral theorem

For a selfadjoint operator A, we will construct:

1. $A \rightsquigarrow$ strongly continuous unitary group $U(t) = \mathrm{e}^{-\mathrm{i}At}$,
2. $\rightsquigarrow$ measureable functional calculus,
3. $\rightsquigarrow$ spectral resolution,
4. $\rightsquigarrow$ spectral measure.

Strongly continuous unitary groups

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Lemma

Let $A \in \mathscr{L}(\mathscr{H})$ be symmetric and $(U(t))_{t\in\mathbb{R}}$ in $\mathscr{L}(\mathscr{H})$ be given by

$$U(t) := \mathrm{e}^{-\mathrm{i}At} := \sum_{n=0}^{\infty} \frac{(-\mathrm{i}A)^n}{n!} t^n,$$

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Proof: Homework ☺.

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Any family $(U(t))_{t\in\mathbb{R}}$ of operators on $\mathscr{H}$ satisfying the properties (1) and (2) of the proceeding Lemma is called a **strongly continuous unitary group (SCUG)**.

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The operator $A : D(A) \longrightarrow \mathscr{H}$ with

$$D(A) := \left\{ \varphi \in \mathscr{H} \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} U(t)\varphi := \lim_{t\to 0} \frac{U(t)\varphi - \varphi}{t} \text{ exists} \right\}$$

and

$$A\varphi := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(t)\varphi \Big|_{t=0}$$

is called the (infinitesimal) **generator** of U.

Strongly continuous unitary groups

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We will write

$$U(t) = \mathrm{e}^{-\mathrm{i}At}.$$

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3. $U(t)\varphi := \lim_{\mu\to\infty} e^{-iA_\mu t}\varphi$ exists for all $\varphi \in \mathscr{H}$, $t \in \mathbb{R}$.
4. $(U(t))_{t\in\mathbb{R}}$ is a SCUG generated by A.

Measureable functional calculus

Let $f \in B(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \text{ is measureable \& bounded}\}$ and A be a selfadjoint operator.

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We will first define this map on the space of Schwartz functions

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \sup_{t\in\mathbb{R}} |t^m f^{(n)}(t)| < \infty \;(n, m \in \mathbb{N}) \right\}.$$

Measureable functional calculus

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Using the Fourier transform, we have for any $f \in \mathcal{S}(\mathbb{R})$:

$$f(x) = \int e^{ixt}\,\hat{f}(t)\,dt \qquad \text{and} \qquad \hat{f}(t) = \frac{1}{2\pi}\int e^{-itx} f(x)\,dx.$$

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So, we aim to define

$$\begin{aligned}\Phi_A(f) = f(A) &= \int e^{-iA(-t)}\, \hat{f}(t)\, dt \\ &= \int U(-t)\, \hat{f}(t)\, dt.\end{aligned}$$

Measureable functional calculus

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For $f \in \mathcal{S}(\mathbb{R})$, consider the bounded & coercive sesquilinearform

$$b_f(\psi, \varphi) = \int \hat{f}(t) \langle \psi, U(-t)\varphi \rangle \, \mathrm{d}t \qquad (\varphi, \psi \in \mathscr{H}).$$

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By the Lax-Milgram theorem, we may then define $\Phi_A(f)$ to be the uniquely determined operator in $\mathscr{L}(\mathscr{H})$ s. t.

$$b_f(\psi, \varphi) = \langle \psi, \Phi_A(f)\varphi \rangle.$$

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Extension Theorem

- Any map $\Phi : \mathcal{S}(\mathbb{R}) \longrightarrow \mathscr{L}(\mathscr{H})$ satisfying the properties (1) to (3) admits a unique extension $\widehat{\Phi} : B(\mathbb{R}) \longrightarrow \mathscr{L}(\mathscr{H})$.

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- If $f_n, f \in B(\mathbb{R})$ s.t. $f_n \overset{\text{b.p.}}{\to} f$, then $\Phi(f_n) \overset{\sigma}{\to} \Phi(f)$.

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Uniqueness

Since $\mathcal{S}(\mathbb{R})$ is dense in $C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : \lim_{|t|\to\infty} f(t) = 0\}$, it is sufficient to extend

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Lemma

$B(\mathbb{R})$ is the smallest superset of $C_0(\mathbb{R})$ being closed under bounded pointwise convergence.

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Suppose $\widehat{\Phi}, \widehat{\Psi} : B(\mathbb{R}) \longrightarrow \mathscr{L}(\mathscr{H})$ are both extensions of Φ.

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Hence, $M = B(\mathbb{R})$.

Existence

A linear functional $L : C_0(\mathbb{R}) \longrightarrow \mathbb{C}$ is said to be **positive** if $L(f) \geq 0$ whenever $f \geq 0$.

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Riesz representation theorem

If $L : C_0(\mathbb{R}) \longrightarrow \mathbb{C}$ is a positive linear funtional, there exists a unique measure $\mu : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathbb{C}$ s. t. $\mu(\mathbb{R}) \leq \|L\| < \infty$ and

$$L(f) = \int f \, \mathrm{d}\mu \qquad (f \in C_0(\mathbb{R})).$$

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For any $\varphi \in \mathscr{H}$, the linear functional

$$L_\varphi : C_0(\mathbb{R}) \longrightarrow \mathbb{C}, \quad f \mapsto \langle \varphi, \Phi(f)\varphi \rangle$$

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By the Riesz representation theorem, we thus find a measure μ_φ s.t.

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defines a bounded, coercive sesquilinearform on $\mathscr{H}$. Lax-Milgram now implies the existence of an operator $\hat{\Phi}(f) \in \mathscr{L}(\mathscr{H})$ s.t.

$$\langle \psi, \hat{\Phi}(f)\varphi \rangle = B(\psi, \varphi) \qquad (\varphi, \psi \in \mathscr{H}).$$

Measurable functional calculus

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Proof: Define $E(\Omega) := \Phi_A(\chi_\Omega)$ for any $\Omega \in \mathfrak{B}(\mathbb{R})$. ■

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For $\varphi \in H$, let $\mu_\varphi : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathbb{C}$ given by

$$\mu_\varphi(\Omega) := \langle \varphi, E(\Omega)\varphi \rangle$$

defines a complex measure on $\mathfrak{B}(\mathbb{R})$, the s.c. **spectral measure**.

Spectral measure

Let A be selfadjoint and E be the spectral resolution of A.

Corollary

For $\varphi \in H$, let $\mu_\varphi : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathbb{C}$ given by

$$\mu_\varphi(\Omega) := \langle \varphi, E(\Omega)\varphi \rangle$$

defines a complex measure on $\mathfrak{B}(\mathbb{R})$, the s.c. **spectral measure**.

Moreover,

$$\mu_\varphi(\mathbb{R}) = \mu_\varphi(\sigma(A)) = \|\varphi\|^2.$$

Theorem

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Let A be selfadjoint with functional calculus Φ_A.

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$$\langle \varphi, f(A)\varphi \rangle = \langle \varphi, \Phi_A(f)\varphi \rangle = \int_{\sigma(A)} f \, \mathrm{d}\mu_\varphi$$

for all $f \in B(\mathbb{R})$ and all $\varphi \in H$.

Proof:

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By means of $(\star)$, we now write

$$f(A) = \Phi_A(f) = \int_{\sigma(A)} f \, \mathrm{d}E = \int_{\sigma(A)} f(\lambda) E(\mathrm{d}\lambda), \qquad f \in B(\mathbb{R}).$$

The Spectral theorem

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by

$$\Phi(f) \left[\Phi_A \left(\frac{1}{1 + |f|} \right) \varphi \right] := \Phi_A \left(\frac{f}{1 + |f|} \right) \varphi.$$

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Theorem (Functional calculus)

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be measureable and A be selfadjoint. Then:

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Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be measureable and A be selfadjoint. Then:

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2. $f(A) := \Phi(f) = \int_{\sigma(A)} f \, \mathrm{d}E = \int_{\sigma(A)} f(\lambda) \, E(\mathrm{d}\lambda)$,
3. For any $\varphi \in D(\Phi(f))$, we have

$$\langle \varphi, \Phi(f)\varphi \rangle = \int_{\sigma(A)} f \, \mathrm{d}\mu_\varphi .$$

The Spectral theorem

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Short Version

$$A = \Phi(\mathrm{id}_{\mathbb{R}}).$$

The Spectral theorem

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Long Version

Any selfadjoint operator A has a unique spectral resolution $E : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{L}(\mathscr{H})$ s.t.

$$A = \int_{\sigma(A)} \lambda E(\mathrm{d}\lambda).$$

Quantum mechanical interpretation

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- The **states** of a quantum system are modelled as normed vectors φ of a Hilbert space $\mathscr{H}$.
- The **observable quantities**, e.g. position, momentum, spin, etc., are modelled as (in general unbounded) selfadjoint operators.
  - **Examples:**

    the momentum operator $p = -i\hbar\nabla$,

    the position operator $(x\varphi)(x) := x\varphi(x)$.

Quantum mechanical interpretation

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If we now **measure** a quantity A at the state φ, the **probability** to find a value within $\Omega \subset \mathbb{R}$ is given by

$$\mu_\varphi(\Omega) = \langle \varphi, E(\Omega)\varphi \rangle.$$

μ_φ is the **probability distribution** of the quantity A at the state φ. Since $\mu_\varphi(\mathbb{R} \setminus \sigma(A)) = 0$, only values in $\sigma(A)$ can be found.

However, the **expectation**

$$\int \lambda \, \mu_\varphi(\mathrm{d}\lambda) = \langle \varphi, A\varphi \rangle$$

can attain any value in $[\inf \sigma(A), \sup \sigma(A)]$.