Karlsruhe Institute of Technology

Institute for Analysis
Prof. Dr. Dirk Hundertmark
Dipl.-Math. Hans-Jürgen Freisinger

Summer Term 2013

Mathematical Physics

2^{nd} Problem sheet

Problem 5

Let $\mathscr{H}$ be a Hilbert space, $A : D(A) \longrightarrow \mathscr{H}$ be a selfadjoint operator and

$$A_\lambda := \frac{\lambda^2}{2}\Big((A+\mathrm{i}\lambda)^{-1} + (A-\mathrm{i}\lambda)^{-1}\Big) \qquad \text{for all } \lambda \in \mathbb{R}.$$

From the lecture, we know that $\lim\limits_{\lambda\to\infty} A_\lambda\varphi = A\varphi$ for all $\varphi \in D(A)$. Moreover the limit

$$U(t)\varphi := \lim_{\lambda\to\infty} \mathrm{e}^{-\mathrm{i}tA_\lambda}\varphi$$

exists for any $\varphi \in \mathscr{H}$ and all $t \in \mathbb{R}$. Prove that for any $\varphi \in D(A)$ and all $s, t \in \mathbb{R}$, we have

a) $\lim\limits_{\lambda\to\infty} \mathrm{e}^{-\mathrm{i}tA_\lambda}\mathrm{e}^{-\mathrm{i}sA_\lambda}\varphi = U(t)U(s)\varphi$,

b) $\lim\limits_{\lambda\to\infty} \int_0^t \mathrm{e}^{-\mathrm{i}sA_\lambda} A_\lambda\varphi \,\mathrm{d}s = \int_0^t U(s)A\varphi\,\mathrm{d}s$.

Problem 6

Let $\mathscr{H}$ be a Hilbert space and $H : D(H) \longrightarrow \mathscr{H}$, $V : D(V) \longrightarrow \mathscr{H}$ be linear operators. Recall that V is said to be *relatively H-bounded* with *bound* $\alpha > 0$ if $D(H) \subset D(V)$ and

$$(1) \qquad \|V\varphi\| \le \alpha\|H\varphi\| + b\|\varphi\| \quad \text{for all } \varphi \in D(H) \text{ with some } b \in (0,\infty).$$

and let $\alpha_0 := \inf\{\alpha > 0 : \text{ (1) holds for some } b \in (0,\infty)\}$.
Show that condition (1) is equivalent to the seemingly stronger condition

$$(2) \qquad \|V\varphi\|^2 \le \tilde{\alpha}^2\|H\varphi\|^2 + \tilde{b}^2\|\varphi\|^2 \quad \text{for all } \varphi \in D(H) \text{ with some } \tilde{b} \in (0,\infty),$$

in the sense that for $\alpha_1 := \inf\{\tilde{\alpha} > 0 : \text{ (2) holds for some } \tilde{b} \in (0,\infty)\}$, we have $\alpha_0 = \alpha_1$.

– please turn over –

Problem 7

For some real function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, consider the multiplication operator

$$M_f : D(M_f) \longrightarrow L^2(\mathbb{R}^d), \;\; \psi \mapsto f\psi$$

with domain

$$D(M_f) := \left\{ \psi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f(x)|^2 |\psi(x)|^2 \, \mathrm{d}x < \infty \right\}$$

and prove:

a) M_f is symmetric.

b) For any $z \in \mathbb{C} \setminus \mathbb{R}$, the operator $R(z) : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ with

$$R(z)\psi := \frac{1}{f - z}\psi \qquad \text{for all } \psi \in L^2(\mathbb{R}^d)$$

is bounded. $R(z)$ is called the *resolvent* of M_f in $z \in \mathbb{C} \setminus \mathbb{R}$.

c) $\operatorname{Ran} R(z) \subset D(M_f)$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.

d) M_f is selfadjoint.

Hint for d): Determine $\operatorname{Ran}(M_f \pm \mathrm{i})$.

The problems are due to discussion in the problem class on Tuesday, May 7, 2013.