Karlsruhe Institute of Technology

Institute for Analysis
Prof. Dr. Dirk Hundertmark
Dipl.-Math. Hans-Jürgen Freisinger

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Mathematical Physics

3rd Problem sheet

Problem 8

Let $\mathscr{H}$ be a Hilbert space and $H_0 : D(H_0) \longrightarrow \mathscr{H}$ be a selfadjoint operator. Moreover, let the operator $V : D(V) \longrightarrow \mathscr{H}$ be a relatively H_0-bounded with relative bound $\alpha_0 < 1$ (compare Problem 6) and define

$$H := H_0 + V \qquad \text{with domain } D(H) = D(H_0).$$

Prove that there exist constants $c, d, \tilde{c}, \tilde{d} > 0$ such that

$$\|H_0\varphi\| \leq c\|H\varphi\| + d\|\varphi\| \qquad \text{and} \qquad \|H\varphi\| \leq \tilde{c}\|H_0\varphi\| + \tilde{d}\|\varphi\|$$

for all $\varphi \in D(H_0)$.

Problem 9 (Fourier transform and Fourier inversion)

In this problem we aim to discover the Fourier inversion formula on the *Schwartz-space*

$$\mathscr{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left| x^\alpha \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| < \infty \text{ for all multiindices } \alpha, \beta \in \mathbb{N}_0^d \right\}$$

which we consider as a linear subspace of $L^2(\mathbb{R}^d)$. The Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ is defined by

$$\hat{f}(k) := \mathscr{F}f(k) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}x\cdot k} f(x)\,\mathrm{d}x \qquad \text{for all } k \in \mathbb{R}^d \text{ and all } f \in \mathscr{S}(\mathbb{R}^d).$$

Moreover, we define the mapping $\mathscr{F}^* : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ by

$$\mathscr{F}^*h(x) := \int_{\mathbb{R}^d} \mathrm{e}^{2\pi\mathrm{i}k\cdot x} h(k)\,\mathrm{d}x \qquad \text{for all } x \in \mathbb{R}^d \text{ and all } h \in \mathscr{S}(\mathbb{R}^d).$$

– please turn over –

Throughout the sequal let $f \in \mathscr{S}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and the Gaussian pulse $g \in \mathscr{S}(\mathbb{R}^d)$ be given by

$$g(x) = \mathrm{e}^{-\pi|x|^2}.$$

a) Show that $\mathscr{F}^*$ is the adjoint operator of $\mathscr{F}$.

b) For $\lambda \neq 0$, define the functions $f_\lambda, f^\lambda \in \mathscr{S}(\mathbb{R}^d)$ by

$$f_\lambda(x) = \frac{1}{\lambda^d} f\left(\frac{x}{\lambda}\right) \qquad \text{and} \qquad f^\lambda(x) = f(\lambda x).$$

Prove that $\mathscr{F} f_\lambda = \hat{f}^\lambda$ and $\mathscr{F}^* \hat{f}^\lambda = (\mathscr{F}^* f)_\lambda$.

c) For $v \in \mathbb{R}^d$, define the operators $T_v, H_v : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ by

$$T_v f(x) := f(x - v) \qquad \text{and} \qquad H_v f(x) := \mathrm{e}^{-2\pi \mathrm{i} x \cdot v} f(x).$$

Show that $\mathscr{F}(T_v f) = H_v \hat{f}$ and $\mathscr{F}(H_v f) = T_{-v} \hat{f}$ and derive similar formulas for $\mathscr{F}^*(T_v f)$ and $\mathscr{F}^*(H_v f)$.

d) Compute the Fourier transform $\hat{g}$ of the Gaussian pulse g.

e) Prove: $f(0) = \lim_{\lambda \to 0} \langle g_\lambda, f \rangle$.

f) Using part a), compute the limit

$$\lim_{\lambda \to 0} \langle \mathscr{F}^* \hat{f}, g_\lambda \rangle.$$

Then using part e), derive a formula representing $f(0)$ in terms of the Fourier transform $\hat{f}$.

g) Use parts c) and f) to give a representation of $f(x)$ in terms of $\hat{f}$ for any $x \in \mathbb{R}^d$.

The problems are due to discussion in the problem class on Tuesday, May 14, 2013.