Karlsruhe Institute of Technology

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# Mathematical Physics

$4^{\text{th}}$ Problem sheet

**Problem 10**

Let $H : D(H) \longrightarrow \mathscr{H}$ be a linear operator on a Hilbert space $\mathscr{H}$. Prove that a subspace $E$ of $\mathscr{H}$ is invariant for $H$ if and only if $E^{\perp}$ is invariant for $A^*$.

**Problem 11**

On some Hilbert space $\mathscr{H}$, consider the linear operators $A$ and $B$ with domains $D(A)$ and $D(B)$. For $z \in \rho(A) \cap \rho(B)$, we define

$$K(z) := (A-z)^{-1} - (B-z)^{-1}.$$

Show that if $K(z_0)$ is a compact operator for some $z_0 \in \rho(A) \cap \rho(B)$, then $K(z)$ is compact for any $z \in \rho(A) \cap \rho(B)$.

**Problem 12**

a) Let $\mathscr{H}$ be a Hilbert space and $H : D(H) \longrightarrow \mathscr{H}$ be a selfadjoint operator on $\mathscr{H}$. Moreover for some unitary operator $U \in \mathscr{L}(\mathscr{H})$, let $\widetilde{H} := U^* H U$. Prove that

$$\sigma(\widetilde{H}) = \sigma(H), \qquad \sigma_d(\widetilde{H}) = \sigma_d(H), \qquad \text{and} \qquad \sigma_{\text{ess}}(\widetilde{H}) = \sigma_{\text{ess}}(H).$$

b) Consider $x$ as multiplication operator on $L^2(\mathbb{R})$. Prove that $\sigma_d(x) = \emptyset$.

**Problem 13**

We consider the Laplacian on $H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Prove that $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$.

**The problems are due to discussion in the problem class on Thursday, June 13, 2013.**