Karlsruhe Institute of Technology

Institute for Analysis
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Summer Term 2013

Mathematical Physics

6^{th} Problem sheet

Problem 16
Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $\varphi_n(x) := \varphi(\frac{x}{n})$ for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Prove that

$$\|\nabla \varphi_n\|_{L^2(\mathbb{R}^d)} \to 0$$

if and only if $d = 1$.

Problem 17
For real parameters $0 < a < b$ let the functions $\varphi_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$\varphi_{a,b}(x) = \begin{cases} 1, & \text{if } |x| < a, \\ \left(\ln \frac{a}{b}\right)^{-1} \ln \frac{x}{b}, & \text{if } a \leq |x| \leq b, \\ 0, & \text{if } |x| > b. \end{cases}$$

Prove that choosing the parameter a suitably, one can achieve that

$$\lim_{b \to \infty} \|\nabla \varphi_{a,b}\|_{L^2(\mathbb{R}^2)} = 0.$$

Problem 18
Let $V \in L^1(\mathbb{R}^2)$ be a negative potential decaying at infinity and satisfying

$$(\star) \qquad \int_{\mathbb{R}^2} V(x)\,\mathrm{d}x < 0.$$

Prove that $\sigma_d(-\Delta + V) = \emptyset$.

Supplementary question: Does this result still hold true if in assumption $(\star)$, one replaces the strong inequality by a weak one?

The problems are due to discussion in the problem class on Friday, June 28, 2013, which will take place at 2 p.m. in room 3A-11.1 in the Allianz building.