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SS 2015
24.06.2015

Numerical Methods

Extra Exercise Sheet

Exercise G/LU1: Write two functions in Matlab which implement the Gauss elimination without (Gauss) and with pivoting (GaussP) for a system of linear equations $Ax = b$. Run the two functions for

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Explain the results.

Exercise G/LU2: Write a Matlab function (LU) which implements the LU-decomposition of a given matrix A. Write another function (LUsolve) which solves $Ax = b$ using the LU-decomposition of the matrix A.
Generate a random 1000×1000 matrix A and 3 random vectors $b1, b2$ and $b3$ of length 1000. Solve the three systems $Ax = b1, Ax = b2, Ax = b3$ using the functions Gauss, GaussP, LUsolve. Compare the times needed to get the solutions.

Exercise CHOL1: Use the Matlab function *chol()* to show that the matrix

$$A = \begin{bmatrix} 229 & 194 & 162 & 244 & 177 \\ 194 & 210 & 105 & 216 & 136 \\ 162 & 105 & 169 & 159 & 125 \\ 244 & 216 & 159 & 330 & 188 \\ 177 & 136 & 125 & 188 & 151 \end{bmatrix}$$

is positive definite, but the matrix

$$B = \begin{bmatrix} 18 & 18 & 16 & 20 & 16 \\ 18 & 22 & 15 & 21 & 17 \\ 16 & 15 & 23 & 26 & 13 \\ 20 & 21 & 26 & 33 & 18 \\ 16 & 17 & 13 & 18 & 15 \end{bmatrix}$$

is not positive definite. Write a Matlab function (cholP) which implements the Cholesky decomposition with diagonal pivoting. Show that the matrix B is positive semi-definite.

Exercise EV1: Write a Matlab function (vonMises) which finds the largest in absolute value eigenvalue and the corresponding eigenvector of a matrix A. Further, write another Matlab function (vonWielandt) which finds the smallest in absolute value eigenvalue and the corresponding eigenvector of a matrix A.

(i) Run the program with the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & -3 & 8 \\ 1 & 0 & 2 \end{bmatrix}.$$

(ii) Find the eigenvalue of A that is closest to -0.5.

(iii) Compare the results with the results given by the Matlab function *eig()*.

Exercise LO1: Use Matlab to produce a diagram of the feasible region for the following linear optimization problem

$$\text{maximize } z = 5x + 2y$$

subject to

$$\begin{aligned} x + y &\leq 15, \\ 4x + 9y &\leq 80, \\ 4x - y &\geq 0, \\ x \geq 0, y &\geq 0. \end{aligned}$$

Use the diagram to find $z_{max} = \max z(x)$.

Exercise LO2: Use the Matlab function *linprog()* to find the solution of the following linear optimization problem

$$\text{maximize } 7x_1 + 9x_2 + 18x_3 + 17x_4$$

subject to

$$\begin{aligned} &2x_1 + 4x_2 + 5x_3 + 7x_4 \leq 42, \\ &x_1 + x_2 + 2x_3 + 2x_4 \leq 17, \\ &x_1 + 2x_2 + 3x_3 + 3x_4 \leq 24, \\ &x_1 \geq 0, \ \ldots, \ x_4 \geq 0. \end{aligned}$$

Note: The Matlab function *linprog(f,A,b)* finds the solution of the linear problem

$$\text{minimize} f(x) \quad \text{subject to} \quad Ax \geq b.$$

Exercise CN1: Let

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}, \qquad b = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0.1 & 0.2 \\ 0.08 & 0.04 & 0 & 0 \\ 0 & -0.02 & -0.02 & 0 \\ -0.01 & -0.01 & 0 & -0.02 \end{bmatrix}, \qquad \Delta b = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.1 \\ -0.1 \end{bmatrix}.$$

Compare the solutions of $Ax = b + \Delta b$ and $(A + \Delta A)x = b$ with the solutions of the unperturbed system $Ax = b$. Explain the results.

Exercise CN2: Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3.999 \end{bmatrix}.$$

How many significant digits of the solution can we trust when we solve $Ax = b$ for an arbitrary vector b? What happens if we change A to

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3.\underbrace{9\ldots\ldots9}_{14 \text{ times}} \end{bmatrix}?$$

How many significant digits of the solution can we trust?
Hint: To get the number of correct significant digits m we have to find the largest integer m for which

$$cond(A) \cdot \epsilon_{\text{mach}} \leq 0.5 \cdot 10^{-m},$$

where ϵ_{mach} is the machine epsilon. In Matlab $\epsilon_{\text{mach}} = 2.2204 \cdot 10^{-16}$.

Exercise NR1: Write a Matlab function (Newton1) which implements the 1d Newton method for finding a solution to a nonlinear equation. Test the program with the following equations

(a) $x^3 - 3x = -2$ with initial guesses -2.4 and 1.2;

(b) $x^3 - x = -3$ with initial guess 0;

(c) $arctan(x) = 0$ with initial guesses 1.4 and 1.39.

Exercise NR2: Write a Matlab function (Newton2) which implements the 2d Newton method for finding a solution to a nonlinear system of equations. Test the program with the following system

$$x\cos(y) + y\cos(x) = 0.9,$$
$$x\sin(y) + y\sin(x) = 0.1.$$