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Partial Differential Equations:
1st problem sheet

Exercise 1: Euler's theorem

A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be positively homogeneous of degree $\alpha \in \mathbb{R}$ if the equation $u(tx_1, \ldots, tx_n) = t^\alpha u(x_1, \ldots, x_n)$ holds for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and all $t > 0$. Prove Euler's theorem:
A function $u \in C^1(\mathbb{R}^n \setminus \{0\})$ is positively homogeneous of degree α if and only if

$$Du(x) \cdot x = \alpha u(x) \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Exercise 2: Separation of variables

Consider the 1-D wave equation $u_{xx} - u_{yy} = 0$ on $\mathbb{R}^2$. Find all solutions $u \in C^2(\mathbb{R}^2)$ of the following forms:

1. sum ansatz: $u(x, y) = f(x) + g(y)$, $f, g \in C^2(\mathbb{R})$.

2. product ansatz: $u(x, y) = f(x)g(y)$, $f, g \in C^2(\mathbb{R})$.

Hint: If a relation of the form $v(x) = w(y)$ holds for all $x, y \in \mathbb{R}$, then v and w are constant.

Exercise 3: Green's formulas and Integration by parts

Let $\Omega \subset \mathbb{R}^n$ denote a domain with piecewise C^1-boundary $\partial\Omega$ and exterior unit normal field ν, let $u, v \in C^2(\bar{\Omega})$. Prove the following identities applying Gauss' divergence theorem to suitable functions:

a) $$\int_\Omega u_{x_i} v \, dx = -\int_\Omega u v_{x_i} + \int_{\partial\Omega} u v \nu_i \, dS$$

b) $\displaystyle\int_\Omega Dv \cdot Du \, dx = -\int_\Omega u\Delta v + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\, dS$ (First Green's formula)

c) $\displaystyle\int_\Omega \big(u\Delta v - \Delta u v\big)\, dx = \int_{\partial\Omega} \big(u\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} v\big)\, dS$ (Second Green's formula)

Verify the equality *b)* considering the functions

i) $u(x,y) = x+y, v(x,y) = xy^2 - y$ on $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$

ii) $u(x,y) = \sqrt{x^2+y^2}, v(x,y) = x+1$ on $\Omega = B_r(0) \subset \mathbb{R}^2$, a disc with radius r in $\mathbb{R}^2$

Exercise 4: Mean value properties

Prove the equality

$$\frac{d}{dr}\Big(\int_{B_r(0)} u(x)\, dx\Big) = \int_{\partial B_r(0)} u(x)\, dS$$

for $u \in C(\mathbb{R}^n)$.
Hint: Use transformation theorem and Fubini's theorem.

Derive the following statements:

a) We set $\alpha(n) = |B_1(0)|$ to be the volume of the unit ball in $\mathbb{R}^n$. Show that $|B_r(0)| = \alpha(n)r^n$ and $\text{area}(\partial B_r(0)) = n\alpha(n)r^{n-1}$.

b) The mean value properties for balls ans spheres are equivalent, i.e.

$$u(x_0) = \fint_{B_r(x_0)} u(x)\, dx \quad \forall r \in (0,R) \quad \Longleftrightarrow \quad u(x_0) = \fint_{\partial B_r(x_0)} u(x)\, dS \quad \forall r \in (0,R)$$