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Partial Differential Equations: 10th problem sheet

Exercise 37: Standing waves

Standing waves are solutions of the n-dimensional wave equation of the form $u(x,t) = v(x)\sin(\omega t + \varphi)$ where $\omega > 0, \varphi \in [0, 2\pi)$ and $v \in C^2(\mathbb{R}^n)$.

a) Find a differential equation for v.

b) Determine all standing waves in dimension 1.

c) Determine all standing waves in dimension 3 that are radially symmetric with respect to the space variable.

Exercise 38: Decay of 3D waves

Let $u \in C^2$ a solution of the 3D wave equation and $U(0) < \infty$ for

$$U(t) := \sum_{\alpha \leq 2} \int_{\mathbb{R}^3} |D^\alpha u(x,t)|\, dx\,.$$

a) Show that there is a constant $K > 0$ independent of u such that $|u(x,t)| \leq \frac{K}{t} U(0)$ for all $t \geq 1$.

b) Applying this result to $v(x,t) = u(x, T-t)$ for some large enough $T > 0$ prove that

$$\lim_{t\to\infty} \frac{U(t)}{t} = 0 \quad \Longrightarrow \quad u \equiv 0$$

Hint: In a) write the solution in the form $u(x,t) = \fint_{\partial B_t(x)} f\nu\, dS$ and apply divergence theorem.

Exercise 39: An explicit 3D wave

Solve the following 3D initial value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } \mathbb{R}^3 \times \mathbb{R}_{>0} \\ u(x,0) &= 0 && \text{on } \mathbb{R}^3 \\ u_t(x,0) &= x_1^2 + x_1 x_2 + x_3^2 && \text{on } \mathbb{R}^3 \end{aligned}$$

Exercise 40: Elastic waves

Consider the 3D elastic wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - c_1^2 \Delta\Big)\Big(\frac{\partial^2}{\partial t^2} - c_2^2 \Delta\Big) u = 0$$

a) Show that the spherical mean M_u satisfies the differential equation

$$\Big(\frac{\partial^2}{\partial t^2} - c_1^2 \frac{\partial^2}{\partial r^2}\Big)\Big(\frac{\partial^2}{\partial t^2} - c_2^2 \frac{\partial^2}{\partial r^2}\Big) v = 0$$

where $v(r,t) = rM_u(r,t)$.

b) Prove that there are functions F_1, F_2, G_1, G_2 in $C^2(\mathbb{R})$ such that

$$v(r,t) = F_1(r + c_1 t) + F_2(r - c_1 t) + G_1(r + c_2 t) + G_2(r - c_2 t)$$

c) Solve the general initial value problem of the 3D elastic wave equation where $v(r,0), v_t(r,0), v_{tt}(r,0), v_{ttt}(r,0)$ are given functions that are sufficiently smooth.

Hint: In b) define a suitable function $U : \mathbb{R}^4 \to \mathbb{R}$ such that $U(z) = u(r,t)$.
In c) it may be convenient to solve the system for $v(r,0) = v_t(r,0) = 0$ first. (Your linear algebra skills are required!)