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Partial Differential Equations: 11th problem sheet

Exercise 41: Energy balance

Let u a solution of the 1D wave equation with initial conditions $u(x,0) = g(x), u_t(x,0) = h(x)$ and $g,h \in C_c^2(\mathbb{R})$. Prove that there is a time $t_0 > 0$ such that the kinetic energy equals the potential energy for $t > t_0$, i.e.

$$\frac{1}{2}\int_{\mathbb{R}} u_t(x,t)^2\,dx = \frac{1}{2}\int_{\mathbb{R}} u_x(x,t)^2\,dx \qquad \forall t > t_0$$

Exercise 42: A kind of maximum principle

In the following let $a < b$ and let $\Delta(a,b)$ denote the closed set enclosed by the triangle with corner points $(2a,0),(2b,0),(a+b,b-a) \in \mathbb{R}\times\mathbb{R}_+$, i.e.

$$\Delta(a,b) = \big\{(x,t) \in \mathbb{R}\times\mathbb{R}_+ : 0 \le t \le b-a,\ 2a+t \le x \le 2b-t\big\}.$$

Let $u_{tt} - u_{xx} \le 0$ in $\Delta(a,b)$ and $u_t(x,0) \le 0$ for $x \in [2a,2b]$.

a) Show for all $a \le a' \le b' \le b$: $\quad u(a'+b', b'-a') \le \frac{1}{2}\big(u(2a',0) + u(2b',0)\big)$

b) Use a) to prove

$$\max_{\Delta(a,b)} u = \max_{[2a,2b]\times\{0\}} u$$

Exercise 43: (Non)linear wave equation I - Abstract setting

In the following let $\|u\| = \sup_{(x,t)\in\mathbb{R}\times[0,T]}|u(x,t)|$ and

$C_b^k(\mathbb{R}\times[0,T]) = \big\{u \in C^k(\mathbb{R}\times[0,T]) :\ \text{All derivatives } D^\alpha u \text{ of order } |\alpha| \le k \text{ are bounded}\big\}$

We define the operator $A : C_b^0(\mathbb{R} \times [0,T]) \to C_b^0(\mathbb{R} \times [0,T])$ by

$$(Af)(x,t) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\, dy\, ds$$

a) Show that $f \in C_b^0(\mathbb{R} \times [0,T])$ implies $Af \in C_b^1(\mathbb{R} \times [0,T])$ and that in this case the following inequalities hold

$$\|Af\| \leq \frac{1}{2} T^2 \|f\| \qquad \|(Af)_t\| \leq T\|f\| \qquad \|(Af)_x\| \leq T\|f\|$$

b) Show that $f \in C_b^1(\mathbb{R} \times [0,T])$ implies $Af \in C_b^2(\mathbb{R} \times [0,T])$ and that in this case Af is the solution of

$$\begin{aligned} u_{tt} - u_{xx} &= f(x,t) && \text{in } \mathbb{R} \times \mathbb{R}_{>0} \\ u(x,0) &= 0 && \text{on } \mathbb{R} \\ u_t(x,0) &= 0 && \text{on } \mathbb{R}\,. \end{aligned}$$

Exercise 44: Nonlinear wave equation II - Result

In this exercise we want to prove the existence of a $C^2(\mathbb{R} \times [0,T])-$solution of the following nonlinear wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= F(u) && \text{in } \mathbb{R} \times \mathbb{R}_{>0} \\ u(x,0) &= 0 && \text{on } \mathbb{R} \\ u_t(x,0) &= 0 && \text{on } \mathbb{R} \end{aligned}$$

for some $T > 0$ where $F \in C^1(\mathbb{R})$ is Lipschitz-continuous with Lipschitz constant L.

a) Use the estimates of exercise 43 to show that there are $R, T > 0$ such that the mapping $u \mapsto A(F(u))$ is a contraction on $U_R := \{u \in C_b^0(\mathbb{R} \times [0,T]) : \|u\| \leq R\}$.

b) Verify the hypothesis' of Banach's fixed point theorem to prove the existence of some $u^* \in C(\mathbb{R} \times [0,T])$ such that $u^* = A(F(u^*))$.

c) Prove $u^* \in C^2(\mathbb{R} \times [0,T])$.

d) Show that u^* solves the problem given above.