Universität Karlsruhe
Institut für Analysis
Dr. Matthias Kurzke
Dipl.-Math.techn. Rainer Mandel

WS 2009/2010

Partial Differential Equations: 13th problem sheet

Exercise 49: Solving the Cauchy problem with complete integrals

In this exercise we will discuss how to solve the Cauchy problem

$$u_x u_y = 1 \qquad\qquad u(0,y) = \log(y) \quad (y > 0)$$

using the complete integral $u(x, y; a, b)$ calculated in exercise 47.

a) Determine functions $a(s), b(s)$ such that $(x, y) \mapsto u(x, y; a(s), b(s))$ solves the initial conditions and the strip condition

$$\frac{d}{ds} \log(s) = Du(0, s, a(s), b(s)) \cdot (0, 1)$$

b) Calculate the envelope $u^*(x, y)$ of this family of solutions.

c) Check that u^* is a solution of the above problem.

Exercise 50: First Order PDE V, Nonuniqueness

Consider the problem

$$u_x^2 + u_y^2 = 1 \qquad\qquad u(\cos(\varphi), \sin(\varphi)) = 0$$

Apply the method of characteristics to determine two solutions of the problem.
Hint: There are two possible choices for the initial data $P(0)$.

Exercise 51: First Order PDE VI

Solve the problem

$$u_x^3 - u_y = 0 \qquad\qquad u(x, 0) = 2x^{3/2} \quad (x > 0)$$

Exercise 52: Picone's example

Let $u \in C^1(\overline{B_1(0)})$ a solution of

$$a(x,y)u_x + b(x,y)u_y = -u \qquad \text{in } B_1(0)$$

and $a(x,y)x + b(x,y)y > 0$ on $\partial B_1(0)$. In the following we write $u(r,\varphi)$ using polar coordinates.

a) Prove that if u attains its maximum in $x_0 \in \partial B_1(0)$ then $\frac{\partial}{\partial r}u(x_0) \geq 0$ and $\frac{\partial}{\partial \varphi}u(x_0) = 0$. What (in)equalities do we obtain for a minimum?

b) Use a) to show $\max_{\overline{B_1(0)}} u \leq 0$ as well as $\min_{\overline{B_1(0)}} u \geq 0$, i.e. $u \equiv 0$.

c) Solve the above problem explicitly for $a(x,y) = x, b(x,y) = cy$ with initial conditions $u(1,\varphi) = f(\varphi)$ for $c = 0$ and $c = 1$ for a given function $f \in C^1([0,2\pi])$.