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Partial Differential Equations: 14th problem sheet

Exercise 53: An implicit solution of a conservation law

Consider the following conservation law

$$u_t - A'(u)Du = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0} \qquad u(x,0) = g(x) \tag{1}$$

for given functions $A \in C^2(\mathbb{R}), g \in C^1(\mathbb{R})$.

a) Verify that if there is some $u \in C^1(\mathbb{R}^n \times \mathbb{R}_{>0})$ that satisfies the implicit formula $u(x,t) = g(x + tA'(u(x,t)))$ then u is a solution of (1) for small times $t > 0$.

b) Use the method of characteristics to obtain this implicit formula.

c) Verify that if $Dg(x + tA'(z)) \cdot A''(z) < \frac{1}{t}$ for all $z \in \mathbb{R}$ then the above implicit formula indeed defines a uniquely determined function u near (x,t).

Exercise 54: Legendre transform

Let H^* denote the Legendre transform of a function $H : \mathbb{R}^n \to \mathbb{R}$.

a) Let $1 < r < \infty$ and $H(p) = \frac{1}{r}|p|^r$. Show that $H^*(q) = \frac{1}{s}|q|^s$ where $\frac{1}{s} + \frac{1}{r} = 1$.

b) Let $A \in \mathbb{R}^{n \times n}$ positive definite, $b \in \mathbb{R}^n$ and $H(p) = \frac{1}{2}pAp + bp$. Compute H^*.

Exercise 55: Minimal surface of revolution

Let $F(p, z) = 2\pi z\sqrt{1+p^2}$, $a, b \in \mathbb{R}$ and I the functional

$$I(u) = \int_0^1 F(u(t), u'(t))\, dt \qquad \text{for } u \in \mathcal{A} := \{v \in C^2[0,1] : v(0) = a, v(1) = b\}$$

that gives the surface of revolution given by the graph of u.
Show that if there exists a minimizer $u \in \mathcal{A}$ then

$$u(x) = \alpha \cosh\big(\frac{x - x_0}{\alpha}\big)$$

for some constants $\alpha, x_0 \in \mathbb{R}$.

Exercise 56: Nonexistence of smooth minimizers

Consider the functional

$$I(u) = \int_{-1}^1 \big(1 - u'(t)^2\big)^2\, dt \qquad \text{for } u \in \mathcal{A} := \{v \in C^1[-1,1] : v(-1) = v(1) = 1\}$$

a) Show that the function $u^*(t) = |t|$ satisfies $I(u^*) = 0$ and satisfies the boundary conditions, but $u^* \notin C^1[-1,1]$. Are there other minimizers of this kind?

b) Prove that there is no minimizer $u \in \mathcal{A}$.

Hint: In b) you may first prove $\inf_{u\in\mathcal{A}} I(u) = 0$ smoothening u^* near zero in order to obtain a sequence of $C^1[0,1]-$functions such that $I(u_n) \to 0$.