Universität Karlsruhe
Institut für Analysis
Dr. Matthias Kurzke
Dipl.-Math.techn. Rainer Mandel

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Partial Differential Equations:
2nd problem sheet

Exercise 5: Laplace equation and separation of variables

a) Determine all solutions of the Laplace equation $\Delta u = 0$ in $(-l, l) \times (-1, 1)$ of the form $u(x, y) = f(x)g(y)$ with $f \in C^2([-l, l]), g \in C^2([-1, 1])$. Which of them satisfy $u(x, -1) = u(x, 1) = 0$ for all $x \in [-l, l]$?

b) Prove that the 2D Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in polar coordinates (r, φ) reads

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

c) Use b) and a suitable separation ansatz to find a solution of $\Delta u = 0$ in $B_1(0)$ with boundary condition

$$u(\cos(\varphi), \sin(\varphi)) = \cos(k\varphi) \qquad \forall \varphi \in [0, 2\pi]$$

for a given constant $k \in \mathbb{N}$.

Exercise 6: Maximum principle for subharmonic functions

Let $U \subset \mathbb{R}^n$ a domain. A function $u \in C(U)$ is called *subharmonic* if

$$u(x_0) \leq \fint_{B_r(x_0)} u(x)\, dS$$

for all $x_0 \in U$ and all $r > 0$ such that $B_r(x_0) \subset U$. Prove that every subharmonic function $u \in C(\overline{U})$ satisfies the maximum principle, i.e. its maximum is attained only on the boundary unless u is constant, whereas the minimum principle in general fails to hold. Give a counterexample.

Exercise 7: Conjugate harmonic functions

Let U a starshaped domain and $v \in C^2(U)$ a harmonic function. Show that there is another harmonic function $w \in C^2(U)$ called *harmonic conjugate to* v such that $u = v + iw$ is holomorphic:

a) Prove first the equivalence of the following statements for a differentiable vector field $\Psi \in C^1(U, \mathbb{R}^2)$:

   i) Ψ is a gradient, i.e. there is a function f such that $Df = \Psi$ on U.

   ii) $(\Psi_1)_y = (\Psi_2)_x$ on U

   iii) For all piecewise C^1 curves $\gamma \subset U$ joining $z_1, z_2 \in U$ the line integral $\int_\gamma \Psi \cdot dx$ is the same.

b) Choose a special vector field Ψ to conclude.

Show that the harmonic conjugate is uniquely determined up to an additive constant.
Hint: Remember the Cauchy-Riemann equations to guess Ψ.

Exercise 8: Mollifiers

A *mollifier* is a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$.

a) Show that the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = e^{-1/x}$ for $x > 0$ and $\psi(x) = 0$ for $x \leq 0$ is in $C^\infty(\mathbb{R})$.

b) Prove the existence of mollifiers.

c) Let φ a mollifier and consider the functions $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. Show that if $f \in C^k(\mathbb{R}^n)$, then for every multiindex α with $|\alpha| \leq k$ and on every compact subset of $\mathbb{R}^n$ we have

$$D^\alpha(\varphi_\varepsilon * f) \implies D^\alpha f \qquad \text{for } \varepsilon \to 0\,,$$

where "$\implies$" denotes uniform convergence.
Is there a suitable condition on $f \in C(\mathbb{R}^n)$ such that $\varphi_\varepsilon * f \implies f$ on $\mathbb{R}^n$?