Universität Karlsruhe
Institut für Analysis
Dr. Matthias Kurzke
Dipl.-Math.techn. Rainer Mandel

WS 2009/2010

Partial Differential Equations: 3rd problem sheet

Exercise 9: 1D Newton potential

Let $I = [a, b]$ an interval, $f \in C(I)$ and $\Phi : I \to \mathbb{R}, z \mapsto -\frac{1}{2}|z|$. Show that the function Φ is the 1D Newton potential, i.e. the function $u := \Phi * f$ is in $C^2(I)$ and solves the Poisson equation $-u'' = f$.

Exercise 10: Maximum principle on unbounded domains

Let $r > 0, U = \mathbb{R}^n \backslash \overline{B_r(0)}$ and $u \in C^2(U) \cap C(\overline{U})$ harmonic such that $\lim_{|x|\to\infty} u(x) = 0$.

a) Prove $\sup_U |u| = \max_{\partial U} |u|$.

b) Generalize this result to a wider class of domains U.

c) Give a counterexample to show that the identity in a) in general fails to hold when the condition $\lim_{|x|\to\infty} u(x) = 0$ is omitted.

Exercise 11: Schwarz reflection principle

Let $U_+ \subset \{x \in \mathbb{R}^n : x_n > 0\}$ a domain such that $U_0 := \partial U_+ \cap \{x_n = 0\} \neq \emptyset$ and let $u \in C^2(U) \cap C(\overline{U})$ harmonic satisfying $u = 0$ on U_0. Show that the function

$$\tilde{u}(x) := \begin{cases} u(x_1, \ldots, x_{n-1}, x_n) & , x \in U_+ \\ 0 & , x \in U_0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & , x \in U_- \end{cases}$$

is harmonic in the interior of $U_+ \cup U_0 \cup U_-$ where $U_- = \{x \in \mathbb{R}^n : (x_1, \ldots, x_{n-1}, -x_n) \in U_+\}$ is the reflected counterpart of U_+ (with respect to x_n).

Exercise 12: Montel's theorem for harmonic functions

Let (u_k) a locally uniformly bounded sequence of harmonic functions in a domain U. Use the gradient estimate and the Arzelà-Ascoli theorem to show that there is a harmonic function u and a subsequence (u_{k_j}) such that on every compact subset of U $u_{k_j} \rightrightarrows u$.
Hints:
i) "Locally uniformly bounded" means that for all $x \in U$ there is an open neighbourhood $V(x) \subset U$ and a constant K_x such that $|u_k(x)| \leq K_x$ for all $k \in \mathbb{N}$ and all $x \in V(x)$.
ii) Consider a countable dense subset of U and use Cantor's diagonal argument.