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Partial Differential Equations: 4th problem sheet

Exercise 13: The Dirichlet problem for Laplace's equation

Solve the Dirichlet problem

$$\Delta u = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+, \qquad u(x,0) = (1 - \frac{|x|}{r}) \cdot 1_{[-r,r]}(x) \quad (x \in \mathbb{R})$$

for any given $r > 0$.

Exercise 14: Removable singularities

A point $x_0 \in U$ is said to be a *removable singularity* of a harmonic function $u : U \setminus \{x_0\} \to \mathbb{R}$ if there is a harmonic function $\tilde{u} : U \to \mathbb{R}$ such that $\tilde{u}(x) = u(x)$ for all $x \in U \setminus \{x_0\}$. Prove:

a) If $u : U \setminus \{x_0\} \to \mathbb{R}$ is harmonic having the property

$$\lim_{x \to x_0} \frac{u(x)}{\Phi(x - x_0)} = 0\,,$$

then x_0 is a removable singularity.

b) If $u : U \setminus \{x_0\} \to \mathbb{R}$ is harmonic such that

$$\lim_{x \to x_0} \frac{u(x)}{\Phi(x - x_0)} = c$$

for $c \in \mathbb{R}$, then $u(x) = c \cdot \Phi(x - x_0) + v(x)$ for some harmonic function v on U.

Hint: Define a candidate $v(x)$ via the Poisson integral formula on a ball around x_0 and show $v(x) = u(x)$ using the maximum principle on annular regions around x_0.

Exercise 15: Fundamental solution for Helmholtz' equation

Consider Helmholtz' equation

$$\Delta u + c^2 u = 0 \qquad \text{in } \mathbb{R}^3$$

for some constant $c > 0$.

a) Find all radially symmetric solutions in $\mathbb{R}^3$.

b) Prove a representation formula similar to the one for Laplace's equation for solutions of the Helmholtz equation in a domain U in terms of the special radial solution

$$\psi(x-y) = -\frac{\cos(c|x-y|)}{4\pi|x-y|}\,.$$

Hint: Use in a) the transformation $w(r) := ru(r)$ to get an ODE with constant coefficients.

Exercise 16: Eigenvalues for Laplace's equation

Prove: If U is a domain and $u : U \to \mathbb{R}$ a function such that

$$\Delta u + \lambda u = 0 \quad \text{in } U, \qquad u|_{\partial U} = 0\,,$$

then we have the following representations:

a) $\lambda \int_U u^2\,dx = \int_U |\nabla u|^2\,dx$ b) $\lambda \int_U u^2\,dx = \frac{1}{2}\int_{\partial U} (x\cdot\nu)\left|\frac{\partial u}{\partial \nu}\right|^2 dS$

Deduce that every such λ for which a nontrivial solution to the above problem exists must be positive.
Hint: Integrate in b) the function $(\Delta u + \lambda u)(Du\cdot x)$.