Universität Karlsruhe WS 2009/2010
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Partial Differential Equations:
5th problem sheet

Exercise 17: Regular points in $\mathbb{R}^2$

Let $U \subset \mathbb{R}^2$ an open domain and $x \in \partial\Omega$. Assume that there is a sufficiently small $R > 0$ and some $\Theta \in (0, 2\pi]$ such that

$$B_R(x) \cap U \subset \{x + (r\cos(\theta), r\sin(\theta)) : 0 < r < R, 0 < \theta < \Theta\}$$

Show that the function ((r, θ) polar coordinates)

$$w(x + (r\cos(\theta), r\sin(\theta)) = -\frac{\log(r)}{\log(r)^2 + \theta^2}$$

is a so-called *weak local barrier*[1] at x, i.e. the following properties hold:

i) w is superharmonic in $U \cap B_R(x) \setminus \{x\}$.

ii) $w > 0$ in $\overline{U} \cap B_R(x) \setminus \{x\}$

iii) $\lim_{z \to x, z \in U} w(z) = 0$.

Exercise 18: A barrier function for C^2−domains

Let $U \subset \mathbb{R}^n$ a C^2−domain, i.e. for each point $x \in \partial U$ there is a neighbourhood $V(x)$ and a twice continously differentiable function $\psi : V(x) \to \mathbb{R}^n$ such that

$$U \cap V(x) = \{x \in V(x) : \psi_n(x) > 0\} \qquad \text{and}$$
$$\partial U \cap V(x) = \{x \in V(x) : \psi_n(x) = 0\}.$$

and $D\Psi_n(x) \neq 0$ in $V(x)$. Determine a barrier function for any given $x \in \partial\Omega$.

[1] One can show that the requirement of a barrier function in Perron's method can be replaced by a weak local barrier.

Exercise 19: A barrier function for convex domains

Let $U \subset \mathbb{R}^n$ a convex domain, $U \neq \emptyset$. Prove that for every $x_0 \in \partial U$ there is a hyperplane $H := \{x \in \mathbb{R}^n : a \cdot x = b\}$ containing x_0 such that $\overline{U} \subset \{a \cdot x \geq b\}$:

i) Prove that for every $y \in \overline{U}^C$ there exists a point $x_y \in \partial U$ such that

$$\inf_{x \in \overline{U}} |y - x| = |y - x_y|$$

and $(x - x_y)(x_y - y) \geq 0$ for all $x \in \overline{U}$.

ii) Show that for every $y \in \overline{U}^C$ there is an $a \in \mathbb{R}^n, a \neq 0$ and $b \in \mathbb{R}$ satisfying

$$a \cdot x \geq b \quad \forall x \in \overline{U} \qquad \text{and} \qquad a \cdot y < b$$

iii) Conclude.

Prove the existence of a barrier function for every $x \in \partial U$.

Exercise 20: Poincaré's "Méthode de Balayage"

Let $U \subset \mathbb{R}^n$ a bounded domain covered by countably many balls $U = \bigcup_{j=1}^{\infty} B_j$ such that $\overline{B_j} \subset U$ for all $j \in \mathbb{N}$. For all $j \in \mathbb{N}$ and all $u \in C(\overline{U})$ denote P_{B_j} the harmonic replacement of u with respect to B_j as defined in the lecture. Let $(i_j)_{j \in \mathbb{N}} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, \ldots)$. Prove:

a) For every subharmonic function $u_0 \in C(\overline{U})$ the inductively defined sequence $u_j := T_{i_j} u_{j-1}$ converges to some harmonic function u_∞.

b) Show that if U is regular domain then u_∞ is the unique solution in $C^2(U) \cap C(\overline{U})$ of the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } U \\ u &= u_0 \quad \text{on } \partial U \end{aligned}$$