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Partial Differential Equations: 6th problem sheet

Exercise 21: Variational formulation of the Neumann Problem

Let $U \subset \mathbb{R}^n$ a C^1-domain and $u \in C^2(\overline{U}), f \in C(\overline{U})$. Show that $I(u) = \inf_{v \in C^2(\overline{U})} I(v)$ for the functional

$$I(v) = \frac{1}{2} \int_U |Dv|^2 \, dx - \int_U f v \, dx$$

if and only if u is the solution of the Neumann problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } U \\ \frac{\partial u}{\partial \nu} &= 0 \qquad \text{on } \partial U \end{aligned}$$

Exercise 22: Variants of the heat equation

Show that the following equations can be transformed into the heat equation:

1. $u_t - a\Delta u + c(t)u = 0$
2. $u_t - a\Delta u + v \cdot Du = 0$
3. $u_t - a\Delta u + b|Du|^2 = 0$

where $a \in \mathbb{R}_{>0}, b \in \mathbb{R}, v \in \mathbb{R}^n$ and $c(t)$ a continuous function on $\mathbb{R}$. Determine in each case a solution formula in terms of the Cauchy data $u(x,0) = g(x)$. Make suitable assumptions on g.
Hint: In c) consider e^{ku}.

Exercise 23: Weierstrass's approximation theorem and the heat kernel

Use the solution formula for the Cauchy problem of the heat equation in one dimension to prove Weierstrass's approximation theorem:
Let $f \in C([a,b])$. Then there is a sequence of polynomials that converges uniformly towards f on $[a,b]$.
Hint: Use the power series representation of the exponential function in the heat kernel.

Exercise 24: Solutions of the heat equation

a) Show that u solves the n-dimensional heat equation in $\mathbb{R}^n \times \mathbb{R}_+$ if and only if for every $\lambda \in \mathbb{R}$ the function $(x,t) \mapsto u(\lambda x, \lambda^2 t)$ is a solution of the heat equation.

b) Use a) to prove that if u is in C^3 and solves the heat equation then the function

$$(x,t) \mapsto x \cdot Du(x,t) + 2tu_t(x,t)$$

is a solution of the heat equation, too.