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Partial Differential Equations: 7th problem sheet

Exercise 25: The heat equation and convexity

a) Let $f : \mathbb{R} \to \mathbb{R}$ a convex function satisfying the growth condition $f(x) \leq Me^{ax^2}$ for given $M, a > 0$. Show that the function

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) f(y)\, dy \qquad (1)$$

is well-defined for sufficiently small $0 < t < t_0$, solves the heat equation and is a convex function in x for all $0 < t < t_0$.

b) Let $f \geq 0$. Show that $t_2 \geq t_1 > 0$ implies $u(x,t_2) \geq u(x,t_1)$ for u given by (1).

Exercise 26: Three Symmetry groups of the heat equation

Show that for any solution $u(x,t)$ of the heat equation the function $u_\varepsilon(x,t)$ defined by

a) $u_\varepsilon(x,t) = u(x, t+\varepsilon)$

b) $u_\varepsilon(x,t) = e^{-\varepsilon x + \varepsilon^2 t} u(x - 2\varepsilon t, t)$

c) $u_\varepsilon(x,t) = \dfrac{1}{\sqrt{1+4\varepsilon t}} \exp\Big(\dfrac{-\varepsilon x^2}{1+4\varepsilon t}\Big) u\Big(\dfrac{x}{1+4\varepsilon t}, \dfrac{t}{1+4\varepsilon t}\Big)$

is still a solution for any given $\varepsilon > 0$ sufficiently small. Show that via a) and c) a constant solution can be transformed into the fundamental solution of the 1D heat equation.

Exercise 27: Weak maximum principle for a parabolic equation

We consider a function $u : U_T \to \mathbb{R}$ for a bounded domain U and $U_T = U \times (0, T]$ with the property

$$u_t - \Delta u + b(x,t) \cdot Du + c(x,t)u \geq 0 \qquad \text{in } U_T$$

Show that if $u \geq 0$ on $\Gamma_T := \partial U \times [0,T] \cup U \times \{0\}$ and $c \geq 0$ in U_T, then $u \geq 0$ on U_T. Conclude that in the case $c \equiv 0$ we have

$$\min_{\overline{U_T}} u = \min_{\Gamma_T} u$$

Exercise 28: A special solution of the 1D heat equation

Determine a bounded solution of the following problem

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) &= 1_{[0,\infty)}(x) && \text{on } \mathbb{R} \times \{0\} \end{aligned}$$