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Partial Differential Equations:
8th problem sheet

Exercise 29: Almost a counterexample to uniqueness

a) Show that the function $u(x,t) := \frac{x}{t\sqrt{t}} e^{-\frac{x^2}{4t}}$ is a solution of

$$\begin{aligned} u_t - u_{xx} &= 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_{>0} \\ \lim_{t \to 0} u(x,t) &= 0 \qquad \forall x \in \mathbb{R} \end{aligned}$$

b) Obviously $v(x,y) \equiv 0$ is another solution to the above problem. Why doesn't this contradict the uniqueness theorem of the lecture for nonnegative functions?

Exercise 30: The maximum principle via energy methods

Let $U_T = U \times (0,T]$ with parabolic boundary Γ_T and let $u \in C^2(\overline{U_T})$ satisfy the differential inequality

$$u_t - \Delta u \leq 0 \qquad \text{in } U_T\,.$$

Prove the maximum principle $\max_{\overline{U_T}} u = \max_{\Gamma_T} u$ via the following steps:

i) $\psi(u) \in C^2(\overline{U_T})$.

ii) $\psi(u)_t - \Delta\psi(u) \leq 0$ in U_T.

iii) $t \mapsto \int_U \psi(u)(x,t)\,dx$ is a nonincreasing function.

iv) $u(x,t) \leq M$ for all $(x,t) \in U_T$.

for $M := \max_{\Gamma_T} u$ and $\psi(u)(x,t) = (u(x,t) - M)_+^4$.

Exercise 31: The heat equation and Burgers' equation

Let $\mu > 0$ and ϑ a solution of the viscous Burgers equation $\vartheta_t + \vartheta\vartheta_x = \mu\vartheta_{xx}$ satisfying

$$\lim_{x\to-\infty} \vartheta(x,t) = \lim_{x\to-\infty} \vartheta_x(x,t) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} |\vartheta(s,t)|\, ds < \infty \quad \text{for all } t > 0\,.$$

a) Show that this equation can be reduced to the heat equation via a transformation of the form

$$u(x,t) = \exp\Big(-\alpha \int_{-\infty}^{x} \vartheta(s,\beta t)\, ds\Big)$$

for $\alpha, \beta \in \mathbb{R}$.

b) Show that the viscous Burgers equation with given initial conditions $\vartheta(x,0) = f(x)$ and $f \in C_c(\mathbb{R})$ is uniquely solvable within the class of functions ϑ mentioned above. Derive a solution formula for ϑ.

Exercise 32: Potentials as limits of temperatures

Let U a bounded domain and $u \in C_1^2(U \times (0,\infty)) \cap C(\overline{U} \times [0,\infty))$ a solution of

$$\begin{aligned} u_t - \Delta u &= f(x,t) && \text{in } U \times (0,\infty) \\ u(x,t) &= g(x,t) && \text{on } \partial U \times (0,\infty) \end{aligned}$$

and $v \in C^2(U) \cap C(\overline{U})$ a solution of

$$\begin{aligned} -\Delta v &= f_0(x) && \text{in } U \\ v(x) &= g_0(x) && \text{on } \partial U \end{aligned}$$

where f, g, f_0, g_0 are given continous functions satisfying

$$\lim_{t\to\infty} \sup_{x\in U} |f(x,t) - f_0(x)| = 0$$
$$\lim_{t\to\infty} \sup_{x\in U} |g(x,t) - g_0(x)| = 0$$

Show that $\lim_{t\to\infty} u(x,t) = v(x)$ uniformly in x.

Hints:
1. Put the statement down to the case $f_0(x) = g_0(x) = 0$ and $v(x) = 0$.
2. In this case: Show that for every $\varepsilon > 0$ there is a time $\tau > 0$ such that for all $t \geq \tau$ we have $w(x,t) \pm u(x,t) \geq 0$ for a function $w(x,t) = (e^r - e^{x_1}) \cdot (\varepsilon + ae^{-\lambda t})$ and suitable constants $r, a, \lambda > 0$. You may use the parabolic maximum principle here.