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WS 2009/2010

Partial Differential Equations:
9th problem sheet

Exercise 33: The heat equation in the half space

Let $g \in C^1(\mathbb{R}_+)$ a function of at most polynomial growth satisfying $g(0) = 0$. solve the initial value boundary value problem

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } \mathbb{R}_{>0} \times \mathbb{R}_{>0} \\ u(x,0) &= 0 && \text{for all } x > 0 \\ \lim_{x \to 0} u(x,t) &= g(t) && \text{for all } t > 0 \end{aligned}$$

and establish the solution formula

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-s)^3}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds$$

via the following steps:

1. Determine the solutions u_ε of the inhomogeneous heat equation $u_t - u_{xx} = \eta_\varepsilon(x) g'(t)$ in $\mathbb{R} \times \mathbb{R}_{>0}$ with $u_\varepsilon(x,0) = 0$ via Duhamel's principle. Derive a representation formula in terms of Φ_{xx}.

2. Now let η_ε a bounded continuous function satisfying $\eta_\varepsilon(x) = 1$ for $x < -\varepsilon$ and $\eta_\varepsilon(x) = -1$ for $x > \varepsilon$. Verify $u(x,t) = \lim_{\varepsilon \to 0} u_\varepsilon(x,t)$.

3. Check that u indeed defines a solution of the problem.

Exercise 34: Radial solutions of the wave equation

Determine all solutions $u(x,t)$ of the 3-dimensional wave equation $u_{tt} - \Delta u = 0$ that are radially symmetric with respect to the space variable $x \in \mathbb{R}^3 \setminus \{0\}$.

Exercise 35: The wave equation in a quadrant

a) Let $\alpha \neq -1$. Find a solution $u(x,t)$ of the one-dimensional wave equation in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ satisfying

$$u(x,0) = g(x) \qquad u_t(x,0) = h(x) \qquad u_t(0,t) = \alpha u_x(0,t)$$

where the functions $g, h \in C^2(\mathbb{R}_+)$ are identically zero near 0.

b) Show that generally no solution to the above problem exists when $\alpha = -1$.

Hint: In a) extend the initial data to $\{x < 0\}$ by some kind of reflection.

Exercise 36: A solution of the inhomogeneous 1D wave equation

Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= x^2 && \text{in } \mathbb{R} \times \mathbb{R}_{>0} \\ u(x,0) &= x && \text{for all } x \in \mathbb{R} \\ u_t(x,0) &= 0 && \text{for all } x \in \mathbb{R}\,. \end{aligned}$$

First guess a solution of the differential equation without considering the boundary conditions, then use d'Alembert's formula to determine the solution.