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Partial Differential Equations

Exercise Sheet 1

Exercise 1

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say, that a function $v \in C^2(\overline{\Omega})$ is *subharmonic*, iff

$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

a) For subharmonic v, show the inequality

$$v(x) \leq \fint_{B_r(x)} v \, dy \quad \text{for all } B_r(x) \subset \Omega.$$

b) Conclude that $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$.

c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Moreover, let u be harmonic and $v := \phi(u)$. Show that v is subharmonic.

d) Again, let u be harmonic. Show that $v := |Du|^2$ is subharmonic.

Exercise 2

a) Let

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto \left(xy, y - \frac{y^2}{2} + \sin x \right).$$

Moreover, let a curve be given by $\Gamma := \{(x, y) : x^2 + y^2 = 1\}$.

Calculate $\int_\Gamma (f \cdot \nu)(x, y) \, d\mu_\Gamma(x, y)$, where ν denotes the outer unit normal with respect to the domain bounded by Γ.

b) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with C^1-boundary. Does the domain

$$K(\Omega) = \{y(x, 1) : x \in \Omega, \, y \in \mathbb{R}^+\} \subset \mathbb{R}^{n+1}$$

also have a C^1 boundary?

Exercise 3

Let $u \in C^0(\mathbb{R}^n)$ satisfy the mean value property in $\mathbb{R}^n$ and assume that $\int_{\mathbb{R}^n} |u|^p \, dx < \infty$ for some $p \in [1, \infty)$. Prove that $u \equiv 0$.

Exercise 4

Suppose $u(x)$ is harmonic in some domain in $\mathbb{R}^n$. Prove that

$$v(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$$

is also harmonic in a suitable domain.