Karlsruhe Institute for Technology (KIT)
Institute for Analysis
Prof. Dr. Tobias Lamm
Dr. Patrick Breuning

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Partial Differential Equations

Exercise Sheet 2

Exercise 1

Let B be a real $n \times n$-matrix, $c \in \mathbb{R}^n$, $Q : \mathbb{R}^n \to \mathbb{R}^n$, $Q(x) = Bx + c$, and $u \in C^2(\mathbb{R}^n)$. Show that

$$-\Delta(u \circ Q) = (L_0 u) \circ Q,$$

where

$$L_0 := -\sum_{i,k=1}^{n} a_{ik} \frac{\partial^2}{\partial x_i \partial x_k},$$

and a_{ik} are the elements of the matrix $A := BB^t$. In particular we have $\Delta(u \circ Q) = (\Delta u) \circ Q$ if B is an orthogonal matrix.

Exercise 2

Determine all functions u on $\mathbb{R}^n \backslash \{0\}$, that are both rotationally symmetric (i.e. $u(x) = f(|x|)$) and harmonic. For that first deduce for $f \in C^2(\mathbb{R} \setminus \{0\})$ the relation

$$\Delta u(x) = f''(r) + \frac{n-1}{r} f'(r) \quad \text{with } r = |x|$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

Exercise 3

Show that the zeros of a real-valued harmonic function are never isolated.

Exercise 4

a) Suppose u is real-valued and harmonic on $B_1(0) \subset \mathbb{R}^2$. For $(x, y) \in B_1(0)$ define

$$v(x, y) = \int_0^y (\partial_1 u)(x, t)\, dt - \int_0^x (\partial_2 u)(t, 0)\, dt.$$

Show that $u + iv$ is holomorphic on $B_1(0)$.

b) Suppose u is a harmonic function on Ω. Prove that the function $x \mapsto x \cdot Du(x)$ is harmonic on Ω.

Exercise 5

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^1-boundary and u is a smooth function on $\overline{\Omega}$ such that $\Delta(\Delta u) = 0$ on Ω and $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, where ν denotes the outer unit normal of Ω. Prove that $u = 0$.