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Partial Differential Equations

Exercise Sheet 6

Exercise 1

a) Let $n = 2$, $0 < R < 1$ and let $u \in C^2(B_R(0) \setminus \{0\}) \cap C^0(\partial B_R(0))$ be harmonic. Assume

$$\frac{u(x)}{\log|x|} \to 0 \qquad \text{for } |x| \to 0.$$

Prove that there is a harmonic extension of u to the ball $B_R(0)$. Proceed as follows:

Let v be a solution of

$$\begin{cases} \Delta v = 0 \text{ in } B_R(0), \\ \quad v = u \text{ on } \partial B_R(0) \end{cases}$$

and let $M := \max_{\partial B_R(0)} |u|$. Moreover let $w := v - u$ and $M_r := \max_{\partial B_r(0)} |w|$ $\forall\, 0 < r \leq R$.

Prove the following:

i) $|v| \leq M$ in $B_R(0)$.

ii) $-M_r \frac{\log|x|}{\log r} \leq w(x) \leq M_r \frac{\log|x|}{\log r} \qquad \forall\, x \in \partial B_r(0) \cup \partial B_R(0)$.

iii) $|w(x)| \leq M_r \frac{\log|x|}{\log r} \qquad \forall\, x \in B_R(0) \setminus B_r(0)$.

iv) $|w(x)| \leq \frac{\log|x|}{\log r} M + \frac{\log|x|}{\log r} \max_{\partial B_r(0)} |u| \qquad \forall\, x \in B_R(0) \setminus B_r(0)$.

v) $w \equiv 0$ in $B_R(0) \setminus \{0\}$.

b) Now let $R > 1$. Let $u \in C^2(\mathbb{R}^2 \setminus B_R(0))$ be harmonic with $u = 0$ on $\partial B_R(0)$ and $\lim\limits_{|x| \to \infty} \frac{u(x)}{\log|x|} = 0$. Show that $u = 0$.

Exercise 2

Let G be the Green function in the ball $B_R(0) \subset \mathbb{R}^2$. Prove the following:

i)
$$G(0, y) = \frac{1}{2\pi}(\log|y| - \log R) \quad \forall\, y \in B_R(0)\backslash\{0\},$$

$$G(x, y) = \frac{1}{2\pi}\left(\log|x - y| - \log\left(\frac{|x|}{R}\left|y - \frac{R^2}{|x|^2}x\right|\right)\right) \quad \forall\, x \in B_R(0)\backslash\{0\} \;\; \forall\, y \in B_R(0)\backslash\{x\}.$$

ii)
$$\frac{\partial G}{\partial \nu_y}(x, y) = \frac{R^2 - |x|^2}{2\pi R\,|x - y|^2} \quad \forall\, x \in B_R(0) \;\; \forall\, y \in \partial B_R(0).$$

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Show that $u \in C^2(\Omega)$ is subharmonic if and only if the following holds:

For every ball $\overline{B} \subset \Omega$ and every harmonic function $w \in C^2(\overline{B})$ with $u \leq w$ on ∂B it holds $u \leq w$ in all of B.