Karlsruhe Institute for Technology (KIT)
Institute for Analysis
Prof. Dr. Tobias Lamm
Dr. Patrick Breuning

WS 2013/2014
12.12.2013

Partial Differential Equations

Exercise Sheet 8

Exercise 1

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $a^{ij} \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ with bounded derivatives. Let the operator

$$Lu := -\sum_{i,j=1}^{n} a^{ij}(x)\partial^2_{ij}u$$

be uniformly elliptic and let $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ be a solution of $Lu = 0$. Show for $v := |Du|^2 + \mu u^2$ and for sufficiently large μ the inequality $Lv \leq 0$ in Ω. Deduce:

$$\|Du\|_{L^\infty(\Omega)} \leq C\left(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\right).$$

Exercise 2

Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded. Let the operator

$$Lu := -\sum_{i,j=1}^{n} a^{ij}(x)\partial^2_{ij}u + \sum_{i=1}^{n} b^i(x)\partial_i u + c(x)u$$

satisfy the relations

$$a^{ij} = a^{ji}, \qquad \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n, \qquad a^{ij}, b^i, c \in C^0(\overline{\Omega}).$$

Let $u \in C^2(\Omega)$ be a subsolution of L, i.e. $Lu \leq 0$. Moreover let $v \in C^2(\Omega)$ be a function with $v > 0$ and $Lv \geq 0$ in Ω. Show that $w := u/v$ does not admit a positive maximum in Ω, except for the case that w is constant.

(Hint: Show that $\sum_{i,j=1}^{n}(a^{ij}\partial^2_{ij}w + (2a^{ij}\partial_j v/v + b^i)\partial_i w) \geq 0$ in $\{w > 0\}$.)

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $S(n)$ the set of symmetric $n \times n$-matrices and $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n))$. We say that $F = F(x,z,p,q)$ is uniformly elliptic in $u \in C^2(\Omega)$, if there exists a $\lambda > 0$ with

$$\sum_{i,j=1}^{n} \frac{\partial F}{\partial q_{ij}}(x, u(x), Du(x), D^2u(x))\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n.$$

For $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ let F be uniformly elliptic in all $tu + (1-t)v$, $0 \leq t \leq 1$, and for fixed $(x,p,q) \in \Omega \times \mathbb{R}^n \times S(n)$ let F be nonincreasing in z. Moreover let $u \leq v$ on $\partial\Omega$ and

$$F(\cdot, u, Du, D^2u) \geq F(\cdot, v, Dv, D^2v) \quad \text{in } \Omega.$$

Show that $u \leq v$ in Ω.