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Partial Differential Equations

Exercise Sheet 9

Exercise 1

Let $1 \leq p < q < \infty$.

a) Show that there exist functions f, g with

$$f \in L^p(\mathbb{R}^n) \setminus L^q(\mathbb{R}^n) \quad \text{and} \quad g \in L^q(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n).$$

b) Let $f \in L^p(\mathbb{R}^n)$ be bounded. Show that $f \in L^q(\mathbb{R}^n)$.

c) Let $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Show that

$$f \in L^s(\mathbb{R}^n) \quad \text{for all } s \in [p, q].$$

Exercise 2

For $k \in \mathbb{N}$ let $f_k : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f_k(x) := \begin{cases} \sin^k(k\pi x) & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that for every $p \in [1, \infty)$ it holds

$$\lim_{k \to \infty} \|f_k\|_{L^p} = 0.$$

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be a measurable set with $\mu(\Omega) = 1$ and let $u : \Omega \to [0, \infty]$ be a measurable function. Determine the limits $p \nearrow \infty$ and $p \searrow -\infty$ of the function

$$\phi(p) = \left(\int_\Omega u^p \, d\mu \right)^{\frac{1}{p}}.$$