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Partial Differential Equations

Exercise Sheet 12

Exercise 1

Show that Theorem VI.4 fails to be true if one replaces $C^{0,\beta}_{\text{loc}}(\mathbb{R}^n)$ by $C^{0,\beta}(\mathbb{R}^n)$.

Exercise 2

For a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,\, r>0} \omega_1(u,x,r) = \sup_{x\in\mathbb{R}^n,\, r>0} \fint_{B_r(x)} |u(y) - u_{x,r}|\, dy.$$

If $[u]_{\mathrm{BMO}(\mathbb{R}^n)} < \infty$, we say that u lies in the space of functions of *bounded mean oscillation* $\mathrm{BMO}(\mathbb{R}^n)$.

Show the following:

a) If $[u]_{\mathrm{BMO}(\mathbb{R}^n)} = 0$, then u is a.e. equal to a constant.

b) $L^\infty(\mathbb{R}^n)$ is contained in $\mathrm{BMO}(\mathbb{R}^n)$ and $[u]_{\mathrm{BMO}(\mathbb{R}^n)} \le 2\|u\|_{L^\infty}$.

c) Suppose that there exists an $A > 0$ such that for all balls B in $\mathbb{R}^n$ there exists a constant c_B such that

$$\sup_B \fint_B |u(y) - c_B|\, dy \le A.$$

Then $u \in \mathrm{BMO}(\mathbb{R}^n)$ and $[u]_{\mathrm{BMO}(\mathbb{R}^n)} \le 2A$.

d) For all u locally integrable we have

$$\frac{1}{2}[u]_{\mathrm{BMO}(\mathbb{R}^n)} \le \sup_{x\in\mathbb{R}^n,\, r>0} \left(\inf_{c\in\mathbb{R}} \fint_{B_r(x)} |u(y) - c|\, dy \right) \le [u]_{\mathrm{BMO}(\mathbb{R}^n)}.$$

e) Show that the function $u(x) = \log|x|$ is in $\mathrm{BMO}(\mathbb{R}^n)$ but not in $L^\infty(\mathbb{R}^n)$.

f) Let $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that $u \in \mathrm{BMO}(\mathbb{R}^n)$ by proving the inequality

$$\fint_{B_r(x)} |u(y) - u_{x,r}|\, dy \le C \left(\int_{\mathbb{R}^n} |Du|^n\, dy \right)^{\frac{1}{n}}.$$

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $p \in [1, \infty)$. Show by contradiction the following version of the Poincaré inequality: There exists a constant $C = C(p, \Omega) < \infty$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } \int_\Omega u(x)\, dx = 0.$$