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Partial Differential Equations

Exercise Sheet 13

Exercise 1

Let $\Omega \subset \mathbb{R}^n$ be open, $u \in L^1_{\text{loc}}(\Omega)$ and $\Omega_\delta = \{x \in \Omega : B_\delta(x) \subset \Omega\}$. For $h \in \mathbb{R}$ with $|h| < \delta$ and $1 \le i \le n$ we define

$$\Delta^h u \in L^1_{\text{loc}}(\Omega_\delta), \quad \Delta^h u(x) = \frac{u(x+he_i) - u(x)}{h} \quad (\text{where } \Delta^h = \Delta^h_i),$$

and furthermore $u^h \in L^1_{\text{loc}}(\Omega_\delta)$, $u^h(x) = u(x+he_i)$.

For $u, v \in L^1_{\text{loc}}(\Omega)$ show the following statements:

i) $\Delta^h(uv) = (\Delta^h u)v + u^h(\Delta^h v)$.

ii) $\int_\Omega (\Delta^h u)v = -\int_\Omega u(\Delta^{-h} v)$, if $\text{spt}(uv) \subset \Omega_\delta$ and the integrals are well-defined (for example $u \in L^p$, $v \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$).

iii) $\Delta^h(\partial_j u) = \partial_j(\Delta^h u)$ for $u \in W^{1,1}_{\text{loc}}(\Omega)$.

iv) $\|\Delta^h u\|_{L^p(\Omega_\delta)} \le \|\partial_i u\|_{L^p(\Omega)}$, if $u \in W^{1,p}(\Omega)$ (where $1 \le p \le \infty$).

Exercise 2 *(Caccioppoli inequality)*

a) Let $u \in C^2(B_R(x_0))$ be harmonic, i.e. $\Delta u = 0$. Show for $a \in \mathbb{R}$:

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \le \frac{C}{R^2} \int_{B_R(x_0) \setminus B_{\frac{R}{2}}(x_0)} |u-a|^2.$$

Hint: Use the test function $\eta^2(u-a)$, where η is an appropriate cutoff function.

b) Now let $u \in C^2 \cap W^{1,2}(\mathbb{R}^n)$. Choose a to be the mean-value of u on the annulus and conclude with the Poincaré inequality

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \le \theta \int_{B_R(x_0)} |Du|^2 \quad \text{for a } \theta = \theta(n) \in (0,1).$$

Now deduce: Every bounded harmonic function on $\mathbb{R}^n$ is constant (Liouville).

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary. Show that $u : \Omega \to \mathbb{R}^n$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(\Omega)$.