Karlsruhe Institute for Technology (KIT)
Institute for Analysis
Prof. Dr. Tobias Lamm
Dr. Patrick Breuning

WS 2013/2014
06.02.2014

Partial Differential Equations

Exercise Sheet 14

(last exercise sheet)

Exercise 1

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with C^1-boundary. Moreover let $a^{ij}, c \in C^\infty(\Omega)$ with $\sum_{i,j}^n a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where $\lambda > 0$ is a constant. Let

$$Lu = -\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B(\cdot,\cdot)$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu \quad \text{for all } x \in \Omega.$$

Exercise 2

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with C^1-boundary. Let $W_0^{2,2}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{2,2}(\Omega)$. We say that a function $u \in W_0^{2,2}(\Omega)$ is a weak solution of the *biharmonic equation*

$$(*) \qquad \begin{cases} \Delta^2 u = f \text{ in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

if

$$\int_\Omega \Delta u \Delta\varphi \, dx = \int_\Omega f\varphi \, dx \qquad \text{for all } \varphi \in W_0^{2,2}(\Omega).$$

Given $f \in L^2(\Omega)$, prove that there exists a unique weak solution of $(*)$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with C^1-boundary. A function $u \in W^{1,2}(\Omega)$ is a weak solution of *Neumann's problem*

$$(**) \qquad \begin{cases} -\Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

if

$$\int_\Omega Du \cdot D\varphi \, dx = \int_\Omega f\varphi \, dx \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Suppose $f \in L^2(\Omega)$. Prove that $(**)$ has a weak solution if and only if $\int_\Omega f \, dx = 0$.