Karlsruhe Institute for Technology (KIT)
Institute for Analysis
Prof. Dr. Tobias Lamm
Dr. Patrick Breuning

SS 2014
17.04.2014

Partial Differential Equations II

Exercise Sheet 1

Exercise 1

Let $I = [0,1]$. Prove the following: The function $u(x) = x^\alpha$ is in $C^{0,\beta}(I)$ if and only if $\beta \leq \alpha$.

Exercise 2

Let $K \subset \mathbb{R}^n$ be compact, $u : K \to \mathbb{R}$ and $0 < \beta < \gamma \leq 1$. Prove the following interpolation inequality:

$$\|u\|_{C^{0,\gamma}(K)} \;\leq\; \|u\|_{C^{0,\beta}(K)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(K)}^{\frac{\gamma-\beta}{1-\beta}}.$$

Exercise 3

Let $n \geq 2$. For a given $\varepsilon > 0$, determine a function $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $u_\varepsilon \not\equiv 0$, with

$$\|\Delta u_\varepsilon\|_{C^0(\mathbb{R}^n)} + \|u_\varepsilon\|_{C^0(\mathbb{R}^n)} \leq \varepsilon \|D^2 u_\varepsilon\|_{C^0(\mathbb{R}^n)}.$$

Hint: Make the ansatz $u_\varepsilon = \eta_\varepsilon h$ for a harmonic quadratic polynomial h, for example $h(x) = x_1^2 - x_2^2$, and a suitable cut-off function η_ε. For example choose a suitable function $\psi \in C^\infty(\mathbb{R})$ with $\psi(t) = 1$ for $t \leq -1$ and $\psi(t) = 0$ for $t \geq 0$, and set $\eta_\varepsilon(x) = \psi(\varepsilon' \log|x|)$ for $x \neq 0$ with a small $\varepsilon' = \varepsilon'(\varepsilon)$, and set $\eta_\varepsilon(0) := 1$.

Exercise 4

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with C^1-boundary. Show that there is a constant $\Lambda < \infty$, such that for all $x, y \in \Omega$ there is a C^1-path γ from x to y with

$$L(\gamma) \leq \Lambda |x - y|,$$

where $L(\gamma)$ denotes the length of γ.

Hint: For $p \in \partial\Omega$ there is an open neighborhood $W \subset \mathbb{R}^n$ and a C^1-diffeomorphism $\phi : W \to B_1(0)$ with

$$\phi(\Omega \cap W) = \mathbb{H}^n \cap B_1(0), \quad \text{where } \mathbb{H}^n = \mathbb{R}^{n-1} \times (-\infty, 0).$$

Conclude that the statement of the exercise is true for the set $\Omega \cap W$. Now choose finitely many points $p_1, \ldots, p_N \in \overline{\Omega}$ and open sets W_j with $\overline{\Omega} \subset \bigcup_{j=1}^N W_j$. Here, W_j is as above if $p_j \in \partial\Omega$, resp. $W_j = B_{\varrho_j}(p_j) \subset\subset \Omega$ if $p_j \in \Omega$. Now conclude the statement on the whole set Ω.

Exercise 5

Let $(a^{ij}) \in \mathbb{R}^{n\times n}$ be symmetric and (strictly) positive definite, $\mathbb{H} = \{x \in \mathbb{R}^n : \langle x, a\rangle < 0\}$ where $|a| = 1$, and let $u \in C^2(\mathbb{H})$ be a solution of

$$\begin{aligned} L_0 u &= \sum_{i,j=1}^{n} a^{ij}\partial^2_{ij} u = 0 \quad \text{in } \mathbb{H} \\ u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\mathbb{H}. \end{aligned}$$

Show: If $|u(x)| \le C|x|^{2+\alpha}$, then u is a quadratic polynomial.

Hint: Without loss of generality we may assume $a^{ij} = \delta_{ij}$ and $\mathbb{H} = \mathbb{R}^{n-1} \times (0,\infty)$. Now define a function v on $\mathbb{R}^n$ by reflecting u, i.e. $v : \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$, $v(x) = v(\xi, x_n)$, with

$$v(\xi, x_n) = \begin{cases} u(\xi, x_n) & \text{for } x_n \ge 0, \\ -u(\xi, -x_n) & \text{for } x_n < 0. \end{cases}$$