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Partial Differential Equations II

Exercise Sheet 2

Exercise 1

a) Let $\Omega \subset \mathbb{R}^n$ be open and convex, and let $\phi : \overline{\Omega} \to \overline{B_1(0)}$ be a $C^{k,\alpha}$-diffeomorphism with inverse mapping $\psi = \phi^{-1} : \overline{B_1(0)} \to \overline{\Omega}$. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of $Lu = f$ on Ω, where L is an elliptic operator given by

$$Lu = \sum_{i,j=1}^{n} a^{ij} \partial_{ij}^2 u + \sum_{i=1}^{n} b^i \partial_i u + cu \quad \text{with } a^{ij}, b^i, c, f \in C^{0,\alpha}(\overline{\Omega}).$$

Derive an equation that is solved by the function $v : B_1(0) \to \mathbb{R}$, $v(y) = u(\psi(y))$.

b) Consider the transformed differential equation in part a). Determine $C^{0,\alpha}$-bounds for its coefficients. Moreover, determine an ellipticity constant (depending on the ellipticity constant of L).

Exercise 2

Let $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$. Consider a solution $u \in C^{k+2,\alpha}(B_1)$ of the equation $Lu = f$ on $B_1 = B_1(0) \subset \mathbb{R}^n$, where

$$Lu = -\sum_{i,j=1}^{n} a^{ij} \partial_{ij}^2 u + \sum_{i=1}^{n} b^i \partial_i u + cu$$

is an elliptic operator with $a^{ij}, b^i, c, f \in C^{k,\alpha}(B_1)$ and $\sum_{i,j=1}^{n} a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2$ for $x \in B_1$, $\xi \in \mathbb{R}^n$. Prove the inner estimate

$$\|u\|_{C^{k+2,\alpha}(B_{\frac{1}{2}})} \leq C(\|f\|_{C^{k,\alpha}(B_1)} + \|u\|_{C^0(B_1)}).$$

Hint: You may use the Schauder estimate. Derive an equation for $\partial_j u$ and apply induction.