Karlsruhe Institute for Technology (KIT)
Institute for Analysis
Prof. Dr. Tobias Lamm
Dr. Patrick Breuning

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Partial Differential Equations II

Exercise Sheet 4

Exercise 1

Prove in Theorem II.3 (Local supremum estimate) the remaining case $p \in (0, \mu)$.

Exercise 2

Let $Lu = \partial_j(a^{ij}\partial_i u)$ be uniformly elliptic with $a^{ij} \in C^1(\mathbb{R}^n)$, $\|a^{ij}\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda < \infty$ and $a^{ij} = a^{ji}$. Let $u \in (W^{1,2} \cap C^2)(\mathbb{R}^n)$ be a solution of $Lu = 0$. Finally, let $M(r) := \max_{|x|=r} u(x)$ and $m(r) := \min_{|x|=r} u(x)$.

Show the following statements:

a) M is nondecreasing and m is nonincreasing.

b) Either u is constant, or there are constants $c_0 > 0$, $\alpha > 0$ and $C < \infty$ with

$$M(r) - m(r) \geq c_0 r^\alpha - C.$$

c) If u is bounded, then u is constant (Theorem of Liouville).

Exercise 3 (Bernstein theorem for minimal graphs)

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a minimal surface, which is given as the graph of the smooth function $u : \mathbb{R}^n \to \mathbb{R}$ (i.e. $u \in C^\infty(\mathbb{R}^n)$ and $\Sigma = \{(x, u(x)) : x \in \mathbb{R}^n\}$).

Show: If the slope of the graph is bounded, then Σ is an n-dimensional affine plane in $\mathbb{R}^{n+1}$.

Hint: As Σ is a minimal surface, u satisfies the equation $\partial_i\left(\left(1 + |Du|^2\right)^{-\frac{1}{2}} \partial_i u\right) = 0$. This can be used without proof. The slope of the graph in the point $(x, u(x))$ is given by $|Du(x)|$.