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Partial Differential Equations II

Exercise Sheet 5

Exercise 1

Consider the function

$$u(x) := \log(1 + |\log|x||) \qquad \text{for } x \in B_1(0) \setminus \{0\} \subset \mathbb{R}^2.$$

Show the following statements:

a) $u \in W^{1,2}(B_1(0)) \setminus L^\infty(B_1(0))$, and moreover $u \notin W^{2,1}(B_1(0))$.

b)
$$\Delta u(x) = \frac{-1}{|x|^2(1 + |\log|x||)^2},$$
hence $\Delta u \in L^1(B_1(0))$.

c) $N(\Delta u) \notin W^{2,1}(B_1(0))$, where N denotes the Newton potential.
(*Hint:* Show that $u - N(\Delta u)$ is harmonic.)

Exercise 2

Consider in polar coordinates the function

$$u(re^{i\varphi}) := r^2 \log(r) \cos(2\varphi) \qquad \text{for } 0 \leq r < 1$$

and show

$$\Delta u(re^{i\varphi}) = 4\cos(2\varphi) \qquad \text{in } B_1(0),$$

hence $\Delta u \in L^\infty(B_1(0))$, but $u \in W^{2,p} \setminus W^{2,\infty}(B_1(0))$.

Exercise 3

Let $\Omega \subset \mathbb{R}^n$ be an open domain, $p > 0$ and f a measurable function with $|f|^p \in L^1(\Omega)$. For $t > 0$ let

$$\mu(t) := \mu(\{x \in \Omega : |f(x)| > t\}).$$

Show

$$\mu(t) \leq t^{-p} \int_\Omega |f|^p,$$

and

$$\int_\Omega |f|^p = p \int_0^\infty t^{p-1} \mu(t)\, dt.$$