Classical Methods for Partial Differential Equations

Preparation to the Exam

Here is a list of possible type of exercises we will put in the exam. Some of these are probably much harder than those in the exam, but I still encourage you to try. There were several crucial typos in the exercises, as I picked them in books, I hope this didn’t discourage you...

First type of exercise: show that a function is Sobolev.

Exercise 1

1. Consider the function
   \[ u(x) = (1 + x^2)^{-\frac{\alpha}{2}} \frac{1}{\log(2 + x^2)}, \]  
   where \( \alpha \in (0, 1) \). Check that \( u \in W^{1,p}(\mathbb{R}) \) for all \( p \in (\frac{1}{\alpha}, \infty] \) and that \( u \notin L^q(\mathbb{R}) \) for any \( q \in [1, \frac{1}{\alpha}] \).

2. For which \( p \in [1, \infty] \) is the function \( u(x) := |x| \) in \( W^{1,p}(-1, 1) \)?

3. For which \( p \in [1, \infty] \) is the function \( u(x) := \log |x| \) in \( W^{1,p}(-1, 1) \)?

4. For which \( p \in [1, \infty] \) is the function \( u(x) := \log |x| \) in \( W^{1,p}(B_1(0)) \), where \( B_1(0) \subset \mathbb{R}^n \) and \( n > 1 \)?

\[ \text{NEW!} \]

Beweis. 1. Explicit computation; 2. Any \( p \in [1, \infty] \) with weak derivative coinciding a.e. with the classical one where it exists; 3. No \( p \in [1, \infty] \), because the weak derivative coinciding a.e. with the classical one is \( 1/x \) which is not in any \( L^p \) space (notice on the contrary that \( \log |x| \) is as integrable as one wishes); 4. Any \( p \in [1, n] \): the weak gradient coincides a.e. with the classical one where it exists, and it is given by \( x/|x|^2 \), which one can show to be in \( L^p \) for every such \( p \) using spherical coordinates.

Second type: Application of Sobolev embeddings, Poincaré inequality... (The one below is harder than the one you will do in the exam)

Exercise 2

Let \( I := (0, 1) \).

1. (HARD) Let \( q \in [1, \infty] \) and \( r \in (1, \infty] \). Prove that for any \( u \in W^{1,r}(I) \)
   \[ \|u\|_{L^\infty} \leq C \|u\|_{W^{1,r}} \cdot \|u\|_{L^q}^{1-a} \]  
   for some constant \( C = C(q, r) \), where \( a \in (0, 1) \) is defined by
   \[ NEW! a = \frac{1 - a}{q} + \frac{a}{r}. \]  
   (Hint: NEW! Write \( G(u(x)) = \int_0^x G'(u(t))u'(t)dt \), where \( G(t) := |t|^{a-1}t \) and \( \alpha := a^{-1} \).)
2. (EASY if you assume the first point) Let \(1 \leq q < p < \infty\) and \(r \in [1, \infty]\). Prove that
\[
\|u\|_{L^p} \leq C \|u\|_{W^{1,r}}^{b} \|u\|_{L^r}^{1-b}
\]
for a constant \(C = C(p, q, r)\), where \(b \in (0, 1)\) is defined by
\[
b\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q} - \frac{1}{p}.
\]
(Hint: Write \(\|u\|_p^p = \int |u|^q |u|^{p-q} \leq \|u\|_q^q \|u\|^{p-q}_\infty\) and use the point above if \(r > 1\).)

3. The previous number 3. was trivial. NEW!

\[\text{Exercise 3}\]

\[\text{Beweis.} 1.\text{ Using the hint we start with } u \in C_c^\infty(\mathbb{R})\text{ and get}\]
\[
|u(x)|^\alpha = |G(u(x))| \leq \int_0^x |G'(u(t))||u'(t)|dt = \int_0^x \alpha|u(t)|^{\alpha-1}|u'(t)|dt,
\]
where we have used that \(G'(t) = \alpha|t|^{\alpha-1}\). Notice that \(x < 1\). Now we rewrite the condition on the integrability exponents as
\[
1 = \frac{1}{aq} + \frac{1}{r} = \frac{1}{q' + \frac{1}{p'}}.
\]
and we apply Hölder’s inequality with respect to these exponents
\[
|u(x)|^\alpha \leq \int_0^1 \alpha|u(t)|^{\alpha-1}|u'(t)|dt \leq \alpha \left(\int_0^1 |u(t)|^{(\alpha-1)\frac{aq}{1-a}}\right)\frac{1-a}{qa}\left(\int_0^1 |u'(t)|^r\right)^{\frac{1}{r}}.
\]
Notice that by definition of \(\alpha\) we have
\[
(\alpha - 1)\frac{aq}{1-a} = q
\]
Take the \(\alpha\)-root, and then the supremum for \(x \in I\) to conclude this case. Finally, use the density by checking the convergence of each piece of the inequality.

2. Using the hint we start with \(u \in C_c^\infty(\mathbb{R})\) and get
\[
\|u\|_p^p \leq \|u\|_q^q \|u\|^{p-q}_\infty.
\]
We use the point above after taking the \(p\)-root and obtain (for a suitable \(C\))
\[
\|u\|_p \leq C \|u\|_{W^{1,r}}^{\frac{q}{p}} \left(\|u\|_{W^{1,r}}^{a} \cdot \|u\|_{W^{1,r}}^{1-a}\right)^\frac{p-q}{p} = C \|u\|_q^\frac{q}{p}(1-a)^\frac{p-q}{p} \|u\|_{W^{1,r}}^{a(p-q)}.
\]
To conclude it’s enough to check that
\[
b = \frac{a(p-q)}{p} = 1 - \left(\frac{q}{p} + (1-a)\frac{p-q}{p}\right).
\]
In order to check it without wasting one entire year, notice that \(a\) satisfies
\[
a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}.
\]
Third type: applications of Lax-Milgram, and deduction of the PDE associated.
1. Check that \( u \mapsto u(0) \) from \( W^{1,2}(0,1) \) into \( \mathbb{R} \) is a continuous linear functional. Deduce that there exists a unique \( v_0 \in W^{1,2}(0,1) \) such that
\[
u(0) = \int_0^1 (u'v_0' + uv_0) \quad \forall u \in W^{1,2}(0,1). \tag{14}
\]

2. (HARD) Assume \( v_0 \in C^2(0,1) \). Show that \( v_0 \) solves a differential equation with suitable boundary condition (Hint: Consider first \( u \in C_0^\infty(0,1) \), and only after \( u \in W^{1,2}(0,1) \) arbitrary to get the boundary conditions).

**Beweis.** 1. By Sobolev’s embedding \( W^{1,2}(0,1) \subset C_0^{0,\frac{1}{2}} \), we know that
\[
|u(0)| \leq \|u\|_{C^{0,\frac{1}{2}}} \leq C \|u\|_{W^{1,2}},
\]
so the functional is continuous. In order to deduce the existence and uniqueness of such \( v_0 \), we prove that the bilinear form
\[
a(u, v) = \int_0^1 (u'v' + uv)
\]
is continuous and coercive on \( W^{1,2} \). The continuity follows from Hölder’s inequality
\[
|a(u, v)| \leq \int_0^1 |u'| |v'| + |uv| \leq \|u'\|_2 \|v'\|_2 + \|u\|_2 \|v\|_2 \leq 2 \|u\|_{W^{1,2}} \|v\|_{W^{1,2}},
\]
where we have consider the norm on \( W^{1,2} \) induced by the standard scalar product (which is exactly \( a! \)). Coercivity is trivial, as \( a(u, u) = \|u\|_{W^{1,2}}^2 \) by our choice of the norm. Applying Lax-Milgram we get \( v_0 \).

2. Choosing \( u \in C_0^\infty(0,1) \) as by the hint, we have \( u(0) = u(1) = 0 \), so we get by integration by parts
\[
0 = u(0) = \int_0^1 (u'v_0' + uv_0) = uv_0' |_0^1 + \int_0^1 u(-v_0'' + v_0) = \int_0^1 u(-v_0'' + v_0). \tag{18}
\]
Therefore, since \( u \) is arbitrary, we can use the fundamental theorem of the calculus of variation (as did several times in the exercise class) to get that \(-v_0'' + v_0 = 0\). This is our differential equation. To get the boundary conditions, we use once again the weak formulation (14), this time choosing \( u \in C^\infty \):
\[
u(0) = \int_0^1 (u'v_0' + uv_0) = uv_0' |_0^1 + \int_0^1 u(-v_0'' + v_0) = u(1)v_0'(1) - u(0)v_0'(0), \tag{19}
\]
which we rearrange as
\[
u(1)v_0'(1) - u(0)v_0'(0) + 1) = 0. \tag{20}
\]
Choosing \( u \) with \( u(0) = 0 \) and \( u(1) \neq 0 \) we deduce \( v_0'(1) = 0 \), whereas if \( u(0) \neq 0 \) we get \( v_0'(0) = -1 \). Resuming we get:
\[
\begin{cases}
-v_0'' + v_0 = 0 \quad \text{on } (0,1); \\
v_0'(0) = -1, \\
v_0'(1) = 0.
\end{cases} \tag{21}
\]

Exercise 4
1. Check that
\[
    a(u, v) := \int_0^1 (u'v' + uv)dt + (u(1) - u(0))(v(1) - v(0))
\]
(22)
is continuous and coercive bilinear form on $W^{1,2}(0, 1)$. Deduce that for any $f \in L^2$ there exists a unique $v_0$ in $W^{1,2}(0, 1)$ such that $a(u, v_0) = \int_0^1 fu$ for any $u \in W^{1,2}(0, 1)$.

2. (HARD) Assume $v_0 \in C^2(0, 1)$. Show that $v_0$ solves
\[
\begin{align*}
    -u'' + u &= f &\text{on } (0, 1); \\
    u'(0) &= u(1) - u(0), \\
    u''(0) &= u(1) - u(0).
\end{align*}
\]
(23)

(Hint: Consider as above, first $u \in C_c^\infty(0, 1)$, and only after $u \in W^{1,2}(0, 1)$ arbitrary to get the boundary conditions).

**Beweis.** 1. By Sobolev’s embedding $W^{1,2}(0, 1) \subset C^{0, \frac{1}{2}}$, we know that
\[
    |u(1) - u(0)| \leq |u(1)| + |u(0)| \leq 2 \|u\|_{C^{0, \frac{1}{2}}} \leq C \|u\|_{W^{1,2}}.
\]
(24)
This means that the bilinear form $a$ introduced is continuous by Hölder’s inequality:
\[
    |a(u, v)| \leq \int_0^1 |u'v'| + |uv| + |u(1) - u(0)||v(1) - v(0)| \leq (2 + C) \|u\|_{W^{1,2}} \|v\|_{W^{1,2}}.
\]
(25)
Coercivity is trivial, as $a(u, u) = \|u\|_{W^{1,2}}^2 + |u(1) - u(0)|^2 \geq \|u\|_{W^{1,2}}^2$. Applying Lax-Milgram we get $v_0$ (the fact that $u \mapsto \int_0^1 fu$ is a continuous functional has been proven in the lecture).

2. Choosing $u \in C_c^\infty(0, 1)$ as by the hint, we have $u(0) = u(1) = 0$, so we get by integration by parts
\[
    \int_0^1 fu = \int_0^1 (u'v'_0 + uv_0)dt + (u(1) - u(0))(v_0(1) - v_0(0)) = \int_0^1 (-v''_0 + v_0)u,
\]
(26)
where the boundary integral vanishes as in the exercise above. Rewrite this as
\[
    \int_0^1 (-v''_0 + v_0 - f)u = 0,
\]
(27)
and use the fundamental theorem of the calculus of variation as above to get that $-v''_0 + v_0 = f$ as required. To get the boundary conditions, we use once again the weak formulation (26), a bit rearranged, this time choosing $u \in C^\infty$:
\[
    0 = \int_0^1 (u'v'_0 + uv_0 - fu)dt + (u(1) - u(0))(v_0(1) - v_0(0)) = \int_0^1 (-v''_0 + v_0 - f)udt + uv'_0 \big|_0^1 \\
    + (u(1) - u(0))(v_0(1) - v_0(0)) = (u(1) - u(0))(v_0(1) - v_0(0)) + (u(1)v'_0(1) - u(0)v'_0(0)).
\]
If we choose $u(0) = u(1) \neq 0$, we get
\[
    v'_0(1) = v'_0(0).
\]
(28)
Therefore the equation reduces to
\[
    (u(1) - u(0))(v_0(1) - v_0(0) - v'_0(0)) = 0,
\]
(29)
thus choosing $u(0) \neq u(1)$ we obtain
\[
    v'_0(0) = v_0(1) - v_0(0).
\]
(30)
This concludes the proof that $v_0$ satisfies (23). □
Fourth type: Maximum principle

**Exercise 5**

**NEW!** Here $\Omega$ is a bounded open set.

1. Use the maximum principle to show that any possible solution $u$ of

$$\begin{cases}
\Delta u = u^2 & \text{on } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (31)$$

is non-positive.

2. (HARD) Use the maximum principle to show that the only solution $u$ (which we assume smooth) of

$$\begin{cases}
\Delta u = u & \text{on } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (32)$$

is $u(x) \equiv 0$. (Hint: study $u$ in the set where it is positive or where it is negative).

**Beweis.**

1. Since $u^2 \geq 0$, we see that $u$ is sub-harmonic $\Delta u \geq 0$, therefore by the maximum principle the maximum of $u$ is attained at the boundary, where the function is zero.

2. First of all, you may notice that you can apply Lax-Milgram, so you have existence and **UNIQUENESS** of a solution, but $u = 0$ is a trivial solution, so it must be the only one and the result is true. The easiest solution is: rewrite the equation as $-\Delta u + cu = f$ with $c = 1 \geq 0$ and $f = 0$, and apply the maximum principle to deduce that max and min of $u$ are bounded by max and min on the boundary (that is, 0) and max and min of $f$ (again, 0) **This is a good solution.** For a solution if we knew that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, we could argue as follows: we know that the minimum of $u$ is achieved in $\bar{\Omega}$. If it is attained in an internal point $x_0 \in \Omega$, then we would have that $u(x) \geq u(x_0) = \Delta u(x_0) \geq 0$, so $u$ is non-negative; otherwise it is attained at a boundary point, where $u$ is identically 0, and thus $u$ is non-negative again; arguing similarly with the maximum, we reach $u \equiv 0$. Finally, I think the hint suggested to consider the equation solved by the positive and negative part of $u$ (which we know are still Sobolev functions):

$$\begin{cases}
\Delta u_+ = u_+ & \text{on } \Omega \\
u_+ = 0 & \text{on } \partial \Omega \\
\end{cases} \quad \text{and} \quad \begin{cases}
\Delta u_- = u_- & \text{on } \Omega \\
u_- = 0 & \text{on } \partial \Omega
\end{cases} \quad (33)$$

Apply the maximum principle twice to deduce that both $u_+$ and $u_-$ are 0. \(\square\)

**Fifth type:** harmonic equation, heat equation, wave equation, symmetries, explicit solutions...

**Exercise 6**

1. Suppose $u$ is a solution to the heat equation. Show that for any function $\varphi : \mathbb{R} \to \mathbb{R}$ smooth and convex, the function $v := \varphi(u)$ satisfies $v_t \leq \Delta v$. Show that for any function $\psi(t, s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth, convex in $s$ and non-increasing in $t$, $w(t, x) := \psi(t, u(t, x))$ satisfies still $w_t \leq \Delta w$. Prove that $z := |Du|^2 + u^2_t$ satisfies $z_t \leq \Delta z$.

2. Consider a radial solution $u : \mathbb{R}^n \to \mathbb{R}$ to the wave equation $u_{tt} = u_{xx}$, that is a function of the form $u(x, t) = v(|x|, t)$. Deduce which is the differential equation satisfied by $v$. Then call NEW! $V(r, t) = r^{n-1}v(r, t)$, and derive a differential equation for $V$. It is not clear to me how to solve explicitly the case $n = 3$.

3. Show that a harmonic function $u$ whose square $u^2$ is harmonic, must be constant.
Beweis. 1. Derivative of composition.
2. After a slightly lengthy calculation you should reach

\[ v_{tt} = v_{rr} + \frac{n-1}{r} v_r. \]  \( (34) \)

This is equivalent to

\[ V_{tt} = V_{rr} - \frac{n-1}{r} V_r + \frac{n-1}{r^2} V. \]  \( (35) \)

It is not clear if one can explicit the solution in dimension 3...