Classical Methods for Partial Differential Equations

3. Exercise Sheet

Exercise 1
Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. Prove that there exists a constant $C$, depending only on $\Omega$, such that

$$\max_{\Omega} |u| \leq C(\max_{\partial\Omega} |g| + \max_{\Omega} |f|),$$

whenever $u$ is a smooth solution of

$$\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = g & \text{on } \partial\Omega.
\end{cases}$$

(Hint: Consider the inequality $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$, where $\lambda := \max_{\Omega} |f|$.)

Exercise 2
Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$, whose boundary is $C^1$. For $v \in C^1(\bar{\Omega})$, we define its Dirichlet energy as

$$E(v) := \frac{1}{2} \int_{\Omega} |Dv|^2(y)dy.$$ (3)

Show that for a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, we have $\Delta u = 0$ in $\Omega$ if and only if $u$ minimizes $E$, that is $E(u) = \min \{E(v) \mid v \in C^1(\bar{\Omega}), v = u \text{ on } \partial\Omega\}$.

Exercise 3
1. In the plane $\mathbb{R}^2$, consider a fixed radius $R \in (0,1)$ and a harmonic function $u \in C^2(B_R(0) \setminus \{0\}) \cap C^0(\partial B_R(0))$. Assume the condition

$$\lim_{|x| \to 0} \frac{u(x)}{\log |x|} = 0.\quad (4)$$

Prove that $u$ admits a harmonic extension defined on the (full) ball $B_R(0)$, using the following steps:

Let $v$ be a solution of

$$\begin{cases}
\Delta v = 0 & \text{in } B_R(0), \\
v = u & \text{on } \partial B_R(0).
\end{cases} \quad (5)$$

Moreover, let us set $M := \max_{\partial B_R(0)} |u|$, $w := v - u$ and $M_r := \max_{\partial B_r(0)} |w|$ for any $r \in (0, R)$. Prove the following

- $|v| \leq M$ in $B_R(0)$;
- $|w(x)| \leq M_r \frac{\log |x|}{\log r}$ for any $x \in \partial B_r(0) \cup \partial B_R(0)$;
\[ |w(x)| \leq M \frac{\log |x|}{\log r} \text{ for all } x \in B_R(0) \setminus B_r(0); \]
\[ |w(x)| \leq M \frac{\log |x|}{\log r} + \frac{\log |x|}{\log r} \max_{\partial B_r(0)} |u| \text{ for any } x \in B_R(0) \setminus B_r(0); \]
\[ w \equiv 0 \text{ in } B_R(0) \setminus \{0\}. \]

2. Now let \( R > 1 \). Consider a harmonic function \( u \in C^2(\mathbb{R}^2 \setminus B_R(0)) \) with \( u = 0 \) on \( \partial B_R(0) \). Assume further that
\[
\lim_{|x| \to \infty} \frac{u(x)}{\log |x|} = 0. \tag{6}
\]
Show that \( u \equiv 0 \).