Classical Methods for Partial Differential Equations

8. Exercise Sheet

Exercise 1
Let \( \Omega \) be the open square \( \{ x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1 \} \). Define the function
\[
u(x) = \begin{cases} 
1 - x_1 & \text{if } x_1 > 0, |x_2| < x_1; \\
1 + x_1 & \text{if } x_1 < 0, |x_2| < -x_1; \\
1 - x_2 & \text{if } x_2 > 0, |x_1| < x_2; \\
1 + x_2 & \text{if } x_2 < 0, |x_1| < -x_2.
\end{cases}
\]
(1)

For which exponents \( p \in [1, \infty) \) does \( u \) belong to \( W^{1,p}(\Omega) \)?

Exercise 2
Let \( B := B_{1/2}(0) \subset \mathbb{R}^2 \) and \( u : B \to \mathbb{R} \) defined as \( u(x) = \log(\log(1/|x|)) \). Show that \( u \) lies in \( W^{1,2}(B) \), but not in \( L^\infty(B) \).

Exercise 3
Let \( \Omega \subset \mathbb{R}^n \) be an open domain, and let \( u \in W^{1,1}_{loc}(\Omega) \). Show that also the functions \( u^+ := \max\{u, 0\} \), \( u^- := -\min\{u, 0\} \) and \( |u| \) lie in \( W^{1,1}_{loc}(\Omega) \), and determine their weak derivatives.

Exercise 4
Let \( \Omega \) be an open domain in \( \mathbb{R}^n \), and let \( u \in W^{1,1}_{loc}(\Omega) \). Show that for \( c \in \mathbb{R} \) the following holds:
\[
Du(x) = 0 \quad \text{for almost all } x \in N_c := \{ x \in \Omega : u(x) = c \}.
\]
(2)